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A Study of the Methods Used by Elementary School Pupils in Solving Problems in Arithmetic

A Dissertation
in Candidacy for the Degree of
Bachelor of Education

Department of Philosophy

By
Milton Ezra LaZerte

1926
#6

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CHAPTER 1

PURPOSE OF THE INVESTIGATION

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Chapter 1.

This investigation is an attempt at diagnosis of difficulties encountered in solving problems in arithmetic. In the testing of attainments in arithmetic more attention has been given to the mechanical phases of the subject than to the solving of problems. The four fundamental processes have been analyzed in detail, the typical errors for each operation have been made known and timed drill devices have been arranged to utilize the information gathered from the analyses. The present study attempts to ascertain why children make errors in solving their problems. The various types of errors are first listed and then each error is studied in detail in an attempt to find the fundamental reason for the deficiency found.

Sometimes when a Subject gets into difficulty in his problem-solving, there results an error in the written or oral solution offered. If this be the case, the record of the performance carries its own evidence. Frequently, Subjects meet difficulties which they surmount or even avoid as they try to solve their problems. Written solutions, whether correct or incorrect, do not record these less-evident defects in the thought processes. In this investigation an effort is made to discover as many of these deficiencies as possible and to find their origin. The investigation is concerned with the isolation and analysis of actual errors made or difficulties experienced, and with

the discovery of deficiencies which cause the difficulties.

CHAPTER 2

METHODS, SUBJECTS AND MATERIALS

Chapter 2.

Method

The method followed throughout this investigation was that of diagnosis of individual work. All investigations were made by the writer who worked with but one Subject at a time. When a problem was given to a Subject he was allowed to proceed with a written solution. As soon as the solution was completed the Subject was questioned regarding the written work whether the same were right or wrong. He was asked to explain the content and the method of solution. If there chanced to be errors in the mechanical work the investigator ascertained whether the Subject could perform the required manipulation of numbers. This determination was made at the close of the question hour. The written work of the Subject was preserved and the investigator recorded the replies given by the Subject to all questions asked. Through both written exercises and oral questioning the investigator attempted to learn as much as possible concerning the Subject's understanding of the problem under discussion, and the amount of effort expended in trying to get this information was entirely uninfluenced by the fact that the Subject might have the correct answer to the written problem.

Subjects

In the early part of the investigation the Subjects chosen were from grades V and VII of McKay Avenue and

Garneau schools respectively, in the City of Edmonton. With the re-opening of the 1926-27 school year the study was resumed with certain Edmonton school children. A definite list of problems was selected and arranged in the order of their grade assignment. Each child began in the series with a problem that was relatively easy for him and continued either to the end of the series or to a point where the difficulty had increased appreciably. The Subjects who served during the 1926-27 year were : Five Subjects from grade III and four Subjects from grade IV of the King Edward School, four Subjects from grade V and five Subjects from grade VI of the McKay Avenue School, five Subjects from grade VII of the Garneau School and a few Subjects from the University of Alberta. The last-named group of Subjects were used merely to get a contrast at certain points between the work of immature and mature Subjects.

The Subjects chosen were not thought to be representative of their classes as the teachers were asked to send those pupils who were having particular trouble with the subject of arithmetic. As their later performance testified, the pupils selected formed grade groups of an ascending order of attainment in the subject studied. This gave an opportunity to study the performance of groups of varying maturity who had profited by different amounts of school experience and instruction.

Materials

The list of problems used in the major part of the investigation was as follows:

1. Dick is 6 years old. John is 10 years old. How much older is John than Dick ?
2. Jack rode 35 miles on his bicycle yesterday forenoon. He rode 29 miles in the afternoon. How many miles did he ride yesterday ?
3. Alice has saved $3.28. How much more must she save before being able to pay $4.95 for a doll ?
4. A train has seats for 220 persons. After 80 men and 94 women have taken seats, how many seats are empty?
5. Find the cost of 7 pounds of candy at 35 cents a pound.
6. If a quart of milk costs 14 cents, find the cost of 1 pint.
7. When 1 hat costs $3.00 how many can be bought for $24.00 ?
8. Alice is reading an 85 page book. She has read 25 pages. How long will it take her to finish it if she reads 10 pages an hour?
9. A car uses 8 gallons of gasolene in going 160 miles. Find how far it will go on 3 gallons of gasolene.
10. The perimeter of a square is 60 feet. Find its area.
11. If wheat is worth $1.40 a bushel, find how much I will get for 2520 pounds of it.
12. At $1.38 per square yard find the cost of linoleum needed for a kitchen floor 9 feet by 12 feet.
13. Find the volume of a sphere whose radius is 12 inches.

The problems chosen for this investigation were selected primarily to test the abilities of pupils of the junior

intermediate grades. They were given to grades Vl and Vll to determine what types of improvement come with increased schooling. The problems used involve integral numbers. The peculiar difficulties that are added to the task of solving a problem when fractional numbers replace integers are very numerous and often they are not inherent in the problem situation itself. If we think of a problem as having a quantitative and a logical aspect, we may agree that in very many instances the introduction of the fraction serves merely to complicate the quantitative phase of the problem. Because they were designed for the intermediate grades, the problems have been limited to those involving integral numbers expressing only a few of the simpler and more common relationships. To cover the problem-solving activities of even the elementary school this list would need to be supplemented by others relating to more complex, integral relations, to common fractional relations, to decimal numbers and to business applications of fractions, decimals and percentage.

The table on the following page shows the number of problems attempted by pupils 1 to 23 inclusive.

Subject	Age	Grade	Problems Attempted
1	8	111	1-2-3-4-5-6-7
2	8	111	1-2-3-4-5-6-7
3	8	111	1-2-3-4-5-6-7
4	9	111	1-2-3-4-5-6-7
5	9	111	1-2-3-4-5-6-7
6	9	1V	1-2-3-4-5-6-7-8
7	14	1V	1-2-3-4-5-6-7-8
8	11	1V	1-2-3-4-5-6-7-8
9	9	1V	1-2-3-4-5-6-7-8
10	11	V	3-4-5-6-7-8-9-10-11
11	13	V	3-4-5-6-7-8-9-10-11
12	12	V	3-4-5-6-7-8-9-10-11
13	13	V	3-4-5-6-7-8-9-10-11
14	11	V1	4-5-6-7-8-9-10-11-12
15	14	V1	4-5-6-7-8-9-10-11-12
16	13	V1	4-5-6-7-8-9-10-11-12
17	11	V1	4-5-6-7-8-9-10-11-12
18	12	V1	4-5-6-7-8-9-10-11-12
19	12	V11	4-5-6-7-8-9-10-11-12
20	12	V11	4-5-6-7-8-9-10-11-12
21	12	V11	4-5-6-7-8-9-10-11-12
22	13	V11	4-5-6-7-8-9-10-11-12
23	12	V11	4-5-6-7-8-9-10-11-12

Table 1 : Problems attempted by various Subjects.

CHAPTER 3

SUMMARY OF DATA

Chapter 3.

The present chapter presents the data regarding the number and distribution of the errors made by the twenty-three Subjects in working through the schedule of problems outlined on page 7 of this report.

There is presented first a classified list of thirty-five types of errors that were made either while the Subjects were writing solutions or answering questions based on the problems. The frequency assigned to each error cannot be truly representative of its frequency in a given group of intermediate grade children, as it is necessarily dependent upon the type of problem that was presented ~~and~~ upon the particular questioning that happened to accompany the solutions. The writer, upon discovering a weakness in the thinking of the Subject, began questioning informally. The subsequent errors were dependent therefore upon the initial errors or deficiencies that chanced to be noted. The frequencies mean only that in this investigation the particular deficiencies occurred as represented.

Table 2 shows the errors that occurred only in the writing of solutions, while table 3 gives the total errors independent of their source. Table 4 gives the distribution of the errors according to the problem in which they occured. Table 5 gives the total errors per grade for each type and the averages deduced from these

data are given in both table and graph form.

The errors that occured in the work of 23 Subjects in a total of 193 problems were classified as to type and frequency. There are several ways in which the errors might be grouped but in this summary they are arranged under four major headings, (A) Those that seemed to be associated with the attitude of the Subject as he approached or worked with the problem, (B) Those related to analysis of the problem, (C) Those primarily concerned with the method of solution, and (D) Those related to generalizations. The following list gives the types of errors together with their frequencies.

Errors of 23 Subjects in Solving 193 Problems

(A) Attitude	Frequency
1. Lack of confidence	6
2. Inattention	12
(B) Analysis	
3. Unsatisfactory wording of problems	5
Reading and Interpretation	
4. Incomplete associations	19
5. Meagreness of associations	76
6. Difficulty with the language parts of the problem	63
7. Reading the numerical parts of the problem	9
8. Grasping mathematical idea of certain expressions	134
9. Lack of business experience	47
10. Confused by large numbers	2
11. Analyzing into knowns and unknowns	69
12. Analysis not carried to subordinate ideas	85
13. Failure to see relationships	242
14. Lack of imagination	21
15. Inaccurate graphical aids	57

(C) Method	Frequency
(a) Selection of the correct method	
16. Incomplete notions concerning the four fundamental operations	170
17. Guessing	52
18. Trying to find familiar forms of solutions	3
19. A type form used but not understood	2
20. Correct procedure obtained from some cue in the problem	12
(b) Applying the method selected	
21. Relationship is understood but it cannot be put in logical form	16
22. Wrong use of equations	6
23. Formulae or necessary facts not known	6
24. Errors connected with denominate numbers	129
25. Confused by the details of writing	2
26. Memory contributes the wrong items	9
27. Led astray by false cues	9
28. Difficulty with reversed order of operations	44
29. Meagre knowledge of the number system	41
30. Items improperly labelled	1
31. Error in copying digits	3
32. Evaluation of results	7
(D) Generalizations	
33. Specific examples are solved but the generalizations are not grasped	115
34. A memorized generalization cannot be applied	1
35. Faulty concept of the generalization	1
Total	1476

Errors in Actual Problem-Solving: 193 Problems

	Subject																	
Error	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1																1		
3	2							1				1						
4											2							
6	1			3	7		5	3				1	1					1
7					1													
8					1							1	1		1			
9											1	1		1				
11					2													
12		3	1	2	1	1	1	1	2	1	1	3	1		1		3	1
13		2	4	4	4	2	1	3	2		4	12	2	2	3		1	1
15												1						1
16	4	3	5	6	2		1	3	1		2							
17	2				1						1	5						
20			1				1						1				1	
21			2										2					
22												1			1	1	2	
23															1			1
24									1		3	2		1	2	1		1
25							1											
26												1						
27			1		1							1						
28	1	1		3	2	2	4					1	3				1	
29	1	3		3	7	1										1	1	
30															1			
31													1					1
32					2					1								
35															1			
	11	12	14	21	31	6	14	11	6	2	14	31	12	4	11	4	9	7

(Continued)

Table 2: Errors made by 23 Subjects in the written solutions of 193 problems.

Table 2 concluded.

Error	Subject 19	20	21	22	23
4				1	
8					1
13		1			
18	1				
22			1		
23	1	1	1		
24		1			1
29			1	2	
	2	3	3	3	2

Total = 233

Deficiencies in Problem-Solving : 193 Problems

Error	Subject 1	2	3	4	5	6	7	8	9	10	11	12	13
1				1					1				
2	3						1	1	2		1		1
3	2							2				1	
4					1						2	1	
5	3	3	3	4	3	1	3	4	4	3	5	2	5
6	2		3	7	9	1	14	6	2	1	1	5	2
7	1				1		4				1	1	
8	7	5	9	5	6		4	4	3	9	10	11	8
9	2	6	1	2		2	1	1			6	5	5
10		1											
11	5	4	5	8	6		5	3	5	1	1	2	4
12	1	6	5	4	5	3	4	6	8	4	7	6	2
13	9	12	15	15	18	8	16	16	17	2	17	25	9
14		1				4	2	1				3	
15	4	2	3	5	3	3	2	3	3	4	1	4	4
16	19	21	22	20	12	2	10	20	12	4	3	5	4
17	7	1	2	7	3	1	7	4	3		3	8	1
20			1		1		1	2			1	1	3
21	2		2						1		1		2
22												1	
23											1		
24	6	6	7	10	9	4	5	8	10	4	8	8	4
25	1						1						
26							1					2	
27			2	1	2					1		1	
28	4	4	1	8	4	2	7	3	2		1	2	4
29	2	4	1	3	10	1		3	1		1	2	
31							1						1
32				1	1					1	1		1
33	10	8	3	8	5	6	3	3	7	4	7	2	6
34	1												
	91	84	85	109	100	38	92	90	81	38	79	98	66

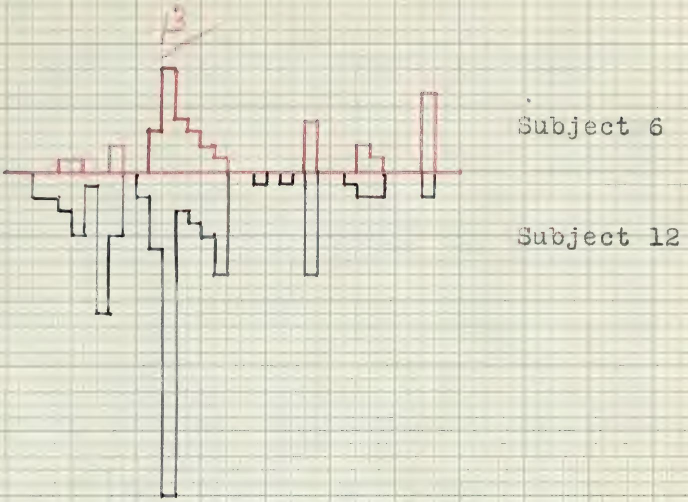
(Continued)

Table 3 : Deficiencies revealed in 193 solutions given by 23 Subjects.

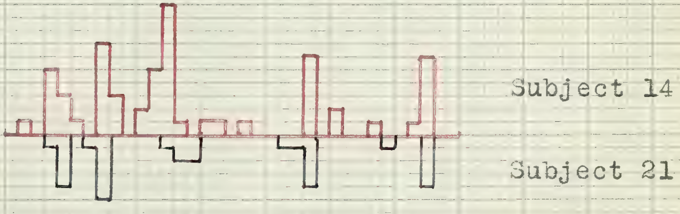
Table 3 concluded

	Subject									
Error	14	15	16	17	18	19	20	21	22	23
1			2						1	
2	1	1	1							
4	5	2	2	2			2	1	1	
5	3	5		6	5	3		4	1	6
6	1			1	4				1	3
7								1		
8	7	5	6	8	8	6	3	5	7	4
9	3	2	2	1	2	3	1		1	
10			1							
11	2		3	4	1	4			2	4
12	5	2	1	8	4	2			1	1
13	10	10	8	11	4	3	1	1	3	1
14	1		1				1	2	3	2
15		1	3	3	4	2	1	2		
16	1	1		1	4	4			4	3
17	1	1		1	2					
18						2				1
19	1	1								
20				3						
21		1	1			1	4			1
22		1	1	2				1		
23		1			1	1	1	1		
24	6	3	5	4	2	4	5	4	6	5
26	2		2	1	1					
27		1				1				
28				1					1	
29	1	1	2	2	1		1	1	3	1
30		1								
31					1					
32	1									
33	6	1	4	7	3	9	2	3	4	4
35		1								
	57	42	45	66	47	45	22	26	39	36

Total errors : 1476



Graph 1 (a): Histogram showing comparative frequency of each of the 35 types of errors for Subjects 6 and 12.



Graph 1 (b): Histogram showing comparative frequency of each of the 35 types of errors for Subjects 14 and 21.

In both sections of the graph the units on the base line represent errots 1 to 35 in succession.

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The following table gives the frequency with which errors occured in the several separate problems.

	Problem														
Error	1	2	3	4	5	6	7	8	9	10	11	12	19	20	TOTAL
1		1		1		1		3							6
2		5		1				3			3				12
3			5												5
4		1				3				5		7	2	1	19
5	2	3	1		23	16	1	10	3	8	5	4			76
6	4	10	10	10	10		7	5	1	2	2	1	1		63
7		1	1	2	3			2							9
8	11	1	2		6	25	10			21	3	5			134
9			2	7	7	5	5	3	4	2	12				47
10					1							1			2
11		4	10	9	17	1	11	6	1	4.	4	2			69
12	1	4	5	22	6	8	3	21	4	4	4	2	1		85
13	4	10	23	16	27	27	38	25	17	11	33	8	3		242
14			1	1	3		1	4			10	1			21
15	1	2	11	15	7	1	3	7	4	4			2		57
16	22	8	9	3	28	73	23	2				2			170
17	1	2	7	7	7	4	7	9	2	1	3		2		52
18									3						3
19									2						2
20				3	2			2	3			2			12
21				1	3	4	2	3	1		1	1			16
22								5				1			6
23									1	4		1			6
24	3	32			13	24		3		24	22	8			129
25	1			1											2
26				3		2	1	2	1						9
27			2	1		2	1		1		2				9
28	17		10	4		2	7	3		1					44
29		1	2	14	4	13	2				4		1		41
30												1			1
31			1								2				3
32			2	3				1					1		7
33	6	6	17	25	9	15	4	9	1	13	9	1			115
34	1														1
35								1							1
	74	91	121	149	176	226	126	129	49	154	119	48	13	1	1476

Table 4 : Distribution of errors according to problems.

Problems 4, 5, 6 and 7 were attempted by the entire 23 Subjects who served throughout the major part of the investigation. It is acknowledged that these Subjects may not be representative of their respective grades, but, in view of the fact that arithmetic was the subject that determined the grade in which they were found , the average number of errors for each type may be indicative in some measure of grade progress with respect to the particular type of error under observation. The following table groups the errors of each type by grades, giving for the four problems, the total number of errors per Subject.

Error	Grade 111	Grade 1V	Grade V	Grade V1	Grade V11
1	0	1	0	0	1
2	0	1	0	0	0
4	0	0	2	1	0
5	13	8	6	6	7
6	9	10	4	2	2
7	1	3	0	0	1
8	23	8	3	4	3
9	8	3	8	2	3
10	1	0	0	0	0
11	20	8	2	2	6
12	14	12	7	5	1
13	52	24	21	9	2
14	1	3	1	0	0
15	10	5	5	4	2
16	68	29	12	7	11
17	16	6	3	0	0
20	1	3	1	0	0
21	3	0	2	1	4
24	18	8	6	0	5
25	1	0	0	0	0
26	00	0	1	5	0
27	3	0	1	0	0
28	6	4	2	1	0
29	17	4	3	3	6
32	1	0	2	0	0
33	12	13	9	9	10

Table 5 : Grade distribution of errors for problems 4, 5, 6 and 7.

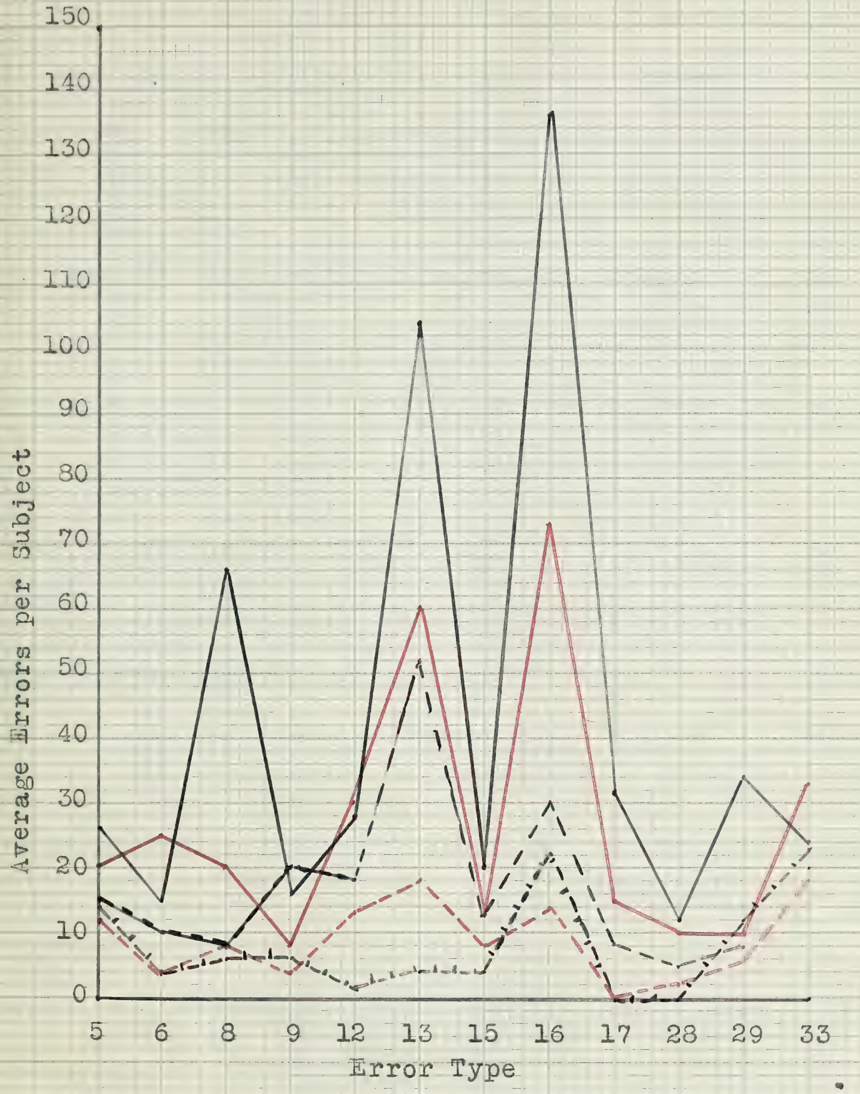
Considering the twelve errors that occur most frequently, the following table gives the average number made per Subject in each grade. The graph which follows the table gives the same information in another form.

Error	Average number of errors per Subject				
	Grade lll	Grade lV	Grade V	Grade Vl	Grade Vll
5	2.6	2.0	1.5	1.2	1.4
6	1.5	2.5	1.0	0.4	0.4
8	6.6	2.0	0.8	0.8	0.6
9	1.6	0.8	2.0	0.4	0.6
12	2.8	3.0	1.8	1.3	0.2
13	10.4	6.0	5.2	1.8	0.4
15	2.0	1.3	1.3	0.8	0.4
16	13.6	7.3	3.0	1.4	2.2
17	3.2	1.5	0.8	0.0	0.0
28	1.2	1.0	0.5	0.2	0.0
29	3.4	1.0	0.8	0.6	1.2
33	2.4	3.3	2.3	1.8	2.0

Table 6 : Frequency of Twelve Specified Errors in each of the Grades Considered.

Conclusions based on the above data :

1. There is a great difference in the number of errors per Subject. Subjects 5 and 12 had 31 errors each while Subjects 10, 19 and 23 had but 2, and in these instances the number problems varied at most by two.
2. The variation in deficiencies is just as great when errors in questioning as well as in solutions are considered. Solutions of equal merit do not necessarily accompany equal degrees of understanding on the part of the Subjects. If the written solution accurately re-



Graph 2: Average Errors per Subject for grades 111, 1V, V, V1 and V11.

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flected the thought processes accompanying the solutions, the total number of deficiencies discovered would be proportional to the number of errors in the written solutions. This is not the case. The ratio of deficiencies to problem errors is as given in the following table .

Subject	Deficiencies	Errors	Ratio
1	91	11	8.3 to 1
4	109	21	5.2 to 1
5	100	31	3.2 to 1
9	81	6	13.5 to 1
10	38	2	19.0 to 1
12	98	31	3.2 to 1
19	45	2	22.5 to 1
23	36	2	18.0 to 1

This must mean that although correct answers can be obtained by certain Subjects, their real efficiency in solving the problem may be low. [illegible]

3. If the ratio of deficiencies to errors be considered for the various grades, it is seen that the average ratio by grades is as follows: Grade 111, 6.0 to 1; Grade 1V, 8.7 to 1; Grade V, 8.3 to 1; Grade V1, 8.7 to 1 and Grade V11, 13.9 to 1. The problems however are not identical, the older Subjects solving problems that the younger Subjects cannot solve. The senior grades are advancing in ability to derive answers more rapidly than in their complete understanding of the problems which they are being called upon to solve.

When problems 4, 5, 6 and 7 only are considered it is found that the average number of deficiencies per grade decreases as 56, 34, 25, 12 and 13. There is improvement in understanding of the elementary problems but not in ability to deal successfully with the more difficult problems. Probably the experiences involved in the earlier problems are noe well under-

stood.

4. Table 4 indicates that the type of error discovered in investigations such as this one depends upon the particular problems chosen. Each particular problem has in the main its own inherent difficulties.

5. Table 6 and Graph 1 show that the greatest relative grade increase in ability is in that related to (1) reading the written problem, (2) understanding the meaning and use of the four fundamental operations, and (3) the ability to grasp relationships buried in the problem.

CHAPTER 4

TYPES OF DIFFICULTIES ILLUSTRATED

Chapter 4

In the previous chapter thirty-five types of errors and deficiencies in problem-solving were named. These errors are discussed immediately below where examples of each are given.

A. 1. Lack of Confidence

If a Subject is lacking in confidence he may not attempt the problem or if he attempts it, his efforts may not be well-directed. Errors due to lack of confidence may be put into two groups, (1) Those where the lack of confidence prevents the Subject from attempting any solution, and, (2) Those where there is a lack of confidence which causes a Subject to change from a correct to an incorrect solution as soon as he is questioned regarding his methods.

Type 1 : Subject 13 in problem 8 after getting a wrong solution was questioned for some time about his work. He was then asked to try a second solution. He sat for 2 minutes and 43 seconds afraid to try again lest he make an error. He was repeatedly urged to proceed. His first attempt was followed by a correct solution.

Type 2 : Subject 4 in problem 2 added 35 and 29. Evidently he was getting his cue from some supplementary source because he changed from the correct to a wrong procedure as soon as he was challenged .

Q.:"Should you add or subtract?"
A.: " Subtract."

The solution was changed from 35 + 29 to 35 - 29.

Subject 22 solved problem 6 correctly by dividing 14 by 2, but, when asked to evaluate his answer, said, " No, I found a mistake. You should multiply by two instead of divide."

Evidently this lack of confidence is found at that level of problem-solving ability when the Subject is not sufficiently able in analysis and evaluation to look back over the work done and find corroborative evidence to justify in his mind the procedure that was followed.

A. 2. Errors Due to Inattention

After discussing the principles that were to be used in problem 2, Subject 1 gave a written solution that was marked by many errors. He repeatedly stated that the boy rode only 35 miles yesterday. Believing that inattention was the factor responsible for the poor work being done, the writer spoke sternly to the Subject and told him that he was not attending to his work. Almost immediately the Subject wrote as follows: " 35 + 29 = 64" and said that the boy rode 64 miles yesterday.

The work of Subject 13 was of this same sort on one occasion. His statements contained many such remarks as, " There are 60 bushels in a pound of wheat", and, " You multiply 108 square feet by 9 to change them to square yards." A sharp reprimend followed. During the remainder of the period the Subject gave good attention and answered without an error a short series of rapid-fire questions that were at least as difficult as those that were previously answered so carelessly.

The Subject who worked problem 8 correctly and a moment later

said that he would add 85 and 25 and then divide by 10, is not interested at that moment in the problem before him.

Comments: If deficiencies 1 or 2 characterize the work, a true measure of ability in problem-solving cannot be obtained. They tend to produce inaccuracies in solutions, inaccuracies that may result at any point of the work. They are in effect ' blanket deficiencies '.

ERRORS THAT RELATE TO ANALYSIS OF PROBLEMS

B. 3. Errors due to the wording of the problem

Two pupils had difficulty with a problem when it was presented in one form but were able to solve it when the form was changed. The two forms of the problem were : (1) A mill owner sold his mill that cost him $13,200 at a loss of $1600 and with the money bought land at $30 per acre. How many acres did he buy? , and, (2) A man bought a mill for $13,200 and when he sold it he lost $1600. He invested the money he got in land at $30 per acre. How many acres did he buy? In the first instance the pupils could not see that the phrase " at a loss of $1600" referred back to the word "sold".

The phrase ' how much more must she save' as found in problem 3 of this investigation led a large number of pupils to treat the problem as a queation in addition. The conversation with Subject 8 was as follows :

Q.: What are you finding?
A.: How much she must save.
Q.: How are you finding it?
A.: By adding a dollar and a few cents on.

Subject 1 understood what the problem meant. His solution was merely " She has to save $1.50".

Q.: Where did you get this $1.50 ?
A.: I put $1.00 with the $3.28 and then put on 50 cents.

When given the example " Subtract 328 from 495" this Subject made the subtraction without error.

Pupil 2 spent 190 seconds in getting an answer $1.63. In explanation she said, " I added $1.00 to $3.28 . This would be $4.28 and she still needs some. Add to find out how much more."

Of like mind was Subject 24 who participated in the following conversation :

Q.: How would you do this question ?
A.: She would have to get $1.00 and -----
Q.: What do you do with the numbers ?
A.: Add them.
Q.: Are you trying to put two numbers together or are you trying to find a difference ?
A.: Add.

Comment:

The findings suggest a problem that merits investigation, namely, ' To what extent would training in the grammatical analysis of arithmetical problems assist pupils in avoiding the difficulties inherent in many poorly-worded problems ?'

B. 4. Word Knowledge of Items Accompanied by Incomplete Concepts.

Real difficulty might be experienced by Subjects 16, 26 and 25 in trying to find the area of a rectangle 8 feet long and 2 1/2 feet wide for it was discovered that their notions regarding a square yard were incomplete. The idea of equivalence of areas was quite absent from their thinking and a square yard was thought to be necessarily square in shape.

The incompleteness of the notion of the fraction 1/9 is evident when a Subject says that if 1/9 of anything be removed 9/8's of it will remain. The pupil who said this was able to find 1/9 of a straight line marked off on the desk.

Subject 17 in problem 12 said that the perimeter is ' all the distance around ', but, he said also that if you wanted to find the amount of linoleum you would have to find the perimeter of the floor. On another occasion this Subject said that the area all around was 186 inches, thus showing that this was his true belief and that he had not merely made a mis-statement.

Subjects 20, 21 and 22 all defined the perimeter as the 'distance around'. Subject 20 said that if the perimeter is 60 feet, the area is 60 x 60 = 3600 square feet; Subject 21 said that the perimeter is equal to length + width; Subject 22 said that it is equal to length x width. [illegible]

Subject 13 deduced from a subdivided figure that one square yard is equal to 9 square feet. He later defined a square yard as the equivalent of 9 square feet, yet almost immediately he proceeded to divide 108 by 3 in an attempt to change 108 square feet to square yards.

Probably in all the cases noted, the Subjects who made the errors could correct them if questioned. The errors are due to faulty organization of items of information. Instead of well-rounded, complete, working concepts, many pupils have verbal definitions which they have not taken the trouble to analyze.

B. 5. Meagreness of Associations

The meagreness of many of the associations which pupils have for certain mathematical terms and notions is illustrated by the responses that were given to certain stimulus words that were given before the problems were attempted.

Besides the scarcity of associations, there is revealed also a great variety in the nature of the associations. For the word 'volume' the following responses were given by the several Subjects indicated by the numbers within the brackets : (40) length, breadth, height- content, (41) book, (42) book, (43) river, book, (44) cubical, rectangular, the form lxaxb, (45) length x width x height x unit of measurement, (46) size, cone, (47) circumference, height, radius, (48) cylinder, (49) l x w x h , something enclosed by straight lines, (52) $4/3 \times 22/7 \times r^2$. For the stimulus word 'area' the following responses were given : (13) length x width, (12) surface covering, (10) surface measure, (11) surface, (16) what it contains in feet, (18) how many acres in the area, (15) perimeter, (19) area of a floor, (20) the top of anything, (21) the part of an arithmetic problem which we use in finding the number of acres or square feet, (22) the number of square rods, and (23) the surface.

The meagreness of the associations is best shown by giving examples of various types so that contrasts may be made. The types were, (1) Those where the wrong idea was held,

(2) Those in which irrelevant responses were offered, (3) Those in which the stimulus term was not defined, (4) Those free from error but not marked by mathematical ideas, (5) Those correct but very special in their application, and (6) Those where the associations were clear and full. Examples of each of these follow.

Type	Subject	Problem	Error
1	17	21	7/8 quart is more than one quart
	28	10	perimeter = area of a field = area =length around
	8	5	10 cents a pound is anything that is 10 cents
2	13	8	an hour : two hours come after it
	11	8	an hour : what hour is it
	12	5	10 cents a pound : I think of 10 cents worth of candy
	11	6	14 cents : one quart costs 14 cents
	14	11	2520 pounds : ounces
	17	10	perimeter : the way you find measurement
3	18	8	85 : there is an 8 and a 5
	12	10	area : area of floor
	23	5	pound : a pound of butter
4	42	13	radius is 12 inches : distance
	43	13	volume : book, river, smoke
	18	8	an hour : used in measuring time
5	22	10	perimeter : how many yards of wire required to go around a field
	18	10	area : how many acres in the area
6	17	8	85 : up near 100
	17	8	an hour : 60 minutes
	40	13	volume : length, breadth, height-content
	40	13	volume of a sphere : 4/3 x 22/7 xr^3
	18	11	bushel of wheat : 4 pecks of wheat : 60 pounds
	16	21	gallon : 4 quarts : 8 pints
	10	5	10 cents a pound : you get one pound for 10 cents
	16	11	2520 pounds : 520 pounds more than a ton

Individual differences exist in the associations which Subjects bring to a problem from their past experience. When each of the following responses was given, the Subject knew that he was about to begin a problem in arithmetic. He had been asked to give for each stimulus word (a) the definition, (b) what the word meant, (c) any image that accompanied the stimulus, and (d) whatever came to mind connected with the word. The following responses show the variation in the wealth of the associations.

Problem 10

		Stimulus word		
Subject	Solution	Area	Square	Perimeter
20	#wrong	means the surface	a figure with four equal sides	length around
19	wrong	I see a wheat field	I see a square piece of wood	A fence around a field
8	wrong	length around field	I don't know	area of a field - breadth, height, width

The 'wrong' under the heading SOLUTION indicates that when the Subject later tried the problem he failed to solve it correctly.

Problem 13

		Stimulus		
Subject	Solution	Volume	Volume of Sphere	Radius is 12 inches
40	Correct	length, breadth, height, content	$4/3 \times 22/7 \times r^3$	1 foot, circle, wheel

45	Wrong	Object with 3 dimensions-space occupied Volume equals l x w x h x unit of measurement	round,solid object ; earth; something about pie r^2	circle; half diam. Diameter is 24 in.- Circ. is 22/7 times diameter.
46	Wrong	size; cone; book	world; rolling object; diameter, radius, circumference	centre to circumference ; circle; diameter;
43	Wrong	river; book; smoke	stars; sun; moon; earth; apple	plate

Problem 6

		Stimulus			
Subject	Solution	Pint	Quart	Costs	14 cents
6	Correct	2 pints equal 1 quart	2 pints 1/4 gal.	Means the price	14 pennies 2 nickels and 4 pennies
21	Correct	A measure for milk or gas	A dry measure used in measuring milk or oil	When finding what you have to pay;	Amount of money used in paying for milk;
11	Wrong	A pint of milk	A quart of milk	What 1 quart of milk costs;	One quart costs 14 cents;
19	Wrong	A pint of berries	A quart of water;	I see the word on things in the store;	Plate;

The above examples indicate :

1. There are great differences in the definiteness of the associations that different Subjects bring to a problem.
2. In general, although not always, those Subjects who are able to solve the problems correctly have the richer and more numerous associations.
3. Some Subjects think quantitatively more readily than others. To some Subjects, 'quart' signifies 2 pints or 1/4 gallon, while to other Subjects it is a measure used for certain purposes, while to yet others it is merely a quart of water or a quart of milk.

One sees certain characteristic differences in the habits of thought of these individuals if their responses for various stimulus words are studied. Consider, for example, the responses of Subjects 6 and 19. The table below gives their responses to several stimulus words as used in this study.

Stimulus	Response of Subject 6	Response of Subject 19
an hour	60 minutes; 1/24 of a day	The car went 90 miles an hour.
85	17 x 5	He got 85 on arithmetic.
Finish	You finish your work	I have finished my work.
To save money	When you get money to keep it;	
Pound	How much it weighs;	A pound of cheese;
10cents a pound	How much things cost	
2520 pounds	25 hundred+ 20 pounds	I see a wagon with over a ton;
Quart	2 pints; 1/4 gallon	A quart of milk;
14 cents	2 nickels and 4 pennies	14 cents worth of bacon
Costs	means the price;	The word 'cost' on a dress;
Forenoon	In the morning;	
Miles	12 blocks to a mile;	A car has gone a mile;
6 years	12 months in a year;	

All responses given by Subjects 6 and 19 are reported above. It is very evident that there are characteristic differences in their respective merits.

A later section of this report will show that there is a parallelism between efficiency in problem-solving and ability to respond with meaningful associations to the stimulus words presented.

B. 6. Difficulties Inherent in Reading

Pupils must find a problem before they can solve it. To find the problem one must be able to read with understanding. The thought must be properly interpreted. Failure to recognize words or inability to interpret the thought will be a handicap. Between reading ability and problem-solving ability we find considerable correspondence. This is due to the fact that we are working with young children who have as yet only immature reading habits. From the records of adult Subjects who served in this investigation, this correspondence is lacking.

Sometimes the sense of a problem is quite disturbed by the mis-reading of a single word. Pupil 4 in problem 3 read 'doll' as 'dollar' and so created for himself this problem: " Alice has saved $3.28. How much more must she save before being able to pay $4.95 for a dollar? ".

Every time there is an error in reading, the difficulty of grasping the thought of the passage is increased. The reading errors are illustrated by giving but a few of each

of the two general types, (a) Those where the error was in word-recognition, and, (b) Those where the difficulty was with the interpretation of the thought.

Errors in Word - Recognition

Subject	Problem	Errors
7	7	hat : had ; many : much
	1	is : has
	3	save : has ; being : begin
	8	is reading : has reading; that : it
3	2	In the questioning of this Subject it was found that he did not know the word 'total'.
4	3	doll : dollar

Difficulty with Thought Interpretation

Pupil 5 in problem 2 increased his difficulty by poor phrasing in the reading. He put the phrase 'yesterday forenoon' with the second sentence instead of leaving it with the first one. The problem was thus made to read,"Jack rode 35 miles on his bicycle. Yesterday forenoon he rode 35 miles ------". In the written solution we find therefore " 35 - 29 = 6."

Subject 7 who in problem 8 wrote " 85 + 25 = 110" and said that there were 110 pages in the book, has not grasped the significance of the 35 as given in the problem.

Subject 6 in problem 11 said that the value of a bushel was 60 pounds.

Subject 4 in problem 3 said that the problem wanted to know how much more than $4.95 Alice had to save.

Subject 4 in problem 2 did not know that he was asked to

find the distance travelled in a whole day, but, thought that the words 'forenoon' and 'afternoon' referred to different days.

Subject 8 in problems 5 and 16 on two different days made the same error. The phrase '7 pounds at 35 cents a pound' and '4 pounds at 30 cents a pound' were interpreted to mean 35 cents for 7 pounds and 4 pounds for 30 cents.

With Subject 3 in problem 5 the following conversation took place:

> Q.: Is 35 cents the cost of 7 pounds or is it the cost of 1 pound?
> A.: Thirty-five cents for 7 pounds of candy.
> Q.: Does the problem say that you get 7 pounds for 35 cents or that you get 1 pound for 35 cents?
> A.: Seven pounds.
> Q.: If you give the storekeeper 35 cents, how much candy will he give you?
> A.: Seven pounds.

In some instances the thought may be correctly gathered at the time of reading, but, interpretation is not sufficiently efficient to ensure that the correct ideas will be held in mind. For example, Subject 17 in problem 12 was asked to find the cost and he estimated that the answer should be about 108 feet. Subject 12 in problem 7 was asked to find 'how many hats' and he estimated the answer to be $72.00.

It should be borne in mind that pupils may read a problem and extract the thought from each part, but the interpretation which they put upon the problem is not the one that was intended by the writer of the problem. The child draws upon his past experience with language and gives to

words the meanings which have been attached to them in his past thinking. One Subject read problem 4 and put his own interpretation upon the words 'have taken'. To him 'take' meant 'to carry to a place'. He therefore solved problem 4 as follows: 220 + 80 + 94 = 394. Naturally if there were 220 seats anywhere and 174 persons take 174 seats along there must be 394 after such an increase. This rather unusual example serves very well to show how very difficult it is to know what is going on in the child's mind as he attempts to solve a problem. Perhaps at many times the teacher is trying diligently to explain the method and the method being explained is not the one that the pupil can adjust to his thinking concerning the problem.

Difficulties in interpretation are sometimes revealed in another manner. No apparent weakness can be discovered, for, although the Subject does not appreciate the full meaning of the content, he has a certain word-knowledge about everything in the problem. A greater measure of analysis can be forced by having the problem re-read. This point was well illustrated in the behavior of Subject 7 in problem 7. His first written solution gave $72.00 instead of the required 8 hats. The Subject was required to re-read the problem orally twelve times. He was then asked to explain the problem, when it was discovered that he had not yet analyzed it. Re-readings thirteen to thirty-two inclusive followed, after which he said, " Add 24 to 3 to get the number of hats". Eight more re-readings were de-

manded after which the Subject said, " I should find how many three's there are in twenty-four". In this instance forty oral readings were required to give the Subject complete understanding of the problem. This Subject's attitude toward problem-solving was wrong. He could probably effect the same results with the expenditure of much less energy.

Subject 8 in problem 8 exemplified the same type of deficiency. A wrong solution was offered after the first reading. At what point in the re-readings the analysis was made was not determined, but, after fourteen oral re-readings, a correct solution was given.

Re-readings before attempting solutions do not have the same effect upon analysis as do readings after erroneous solutions. Many Subjects, among whom were included 7 and 8, were required to re-read certain problems several times before solving, but, correct solutions did not come with the first trial. Problem 7 reported on above was re-read by Subject 7 four times before the first solution was attempted. Probably forty re-readings would have forced analysis, but the evidence favors the view that the attempts at solutions prepare for the later analysis by raising certain doubts and questions in the mind of the Subject, and by convincing him that he has not gathered the full meaning from the written problem.

At the age-level characteristic of the intermediate grades, those Subjects for whom problem-solving presents the fewest difficulties are those in whom reading habits

are most mature. There is a correlation between rate of reading and correctness of solutions. Those pupils who at this age read most rapidly, also solve the problems most successfully. Those problems which by a given pupil can be read most fluently are, on the average, solved more successfully than are those which present greater difficulty in reading.

B. 7. Difficulty in Reading the Numerical Parts of the Problem.

An error in reading may be due to the numerical part of the written statement. This error may be corrected during later re-readings or it may go uncorrected. In the former cases it has no direct bearing upon the written answer, but in the latter cases it affects the solution.

Simple errors of the first sort were :

Subject	Problem	Error
7	4	Read 80 as 8
		Read 220 as 200
4	4	Read 220 as 22
1	5	Read 12367 as one hundred and twenty-three ------
7	3	Read \$3.28 as \$3.20
7	5	Read 12367 as twenty-three hundred and sixty-seven
21	5	Read 12367 as twelve hundred and sixty-seven

An example of an error of the second sort is taken from the work of one Subject in problem 5. This Subject read 35 as 53, did not correct the error on the second reading and used 53 cents in his written solution.

Former investigations show that pupils tend to read

problems twice, once for the words and a second time for the digits. Subjects who slight the digits on the first reading may make many errors of the first type named above.

B. 8. Difficulty in Grasping the Mathematical Idea of Certain Words or Expressions

This deficiency appeared sometimes in the solutions of the problems but most frequently in the oral questioning. In the solutions of the problems the following words and expressions gave difficulty: area, perimeter, cube of radius, 45 cents a pound and square yard. In the oral questioning difficulties arose in connection with each of the following: cube of the radius, perimeter, seven-eighths, area, square (rectilineal figure),minus, sum, difference, plus, 45 cents a pound, total, square root of a number, square of a number, diagonal of a square, diameter and gain.

It would be very surprising if errors such as the above had not been found. These words and expressions are used technically and sometimes they have arbitrary meanings in their mathematical setting. For example, the meaning of 'square yard' is something more than that conveyed by the component ideas of square and yard, because certain ideas of equivalence are introduced. Similarly the 'cube of a number' cannot be derived directly from the meaning of the simple word 'cube'. The expression '45 cents a pound' and the omission of 'for' caused worry to more than one pupil.

The meaning of the simple word 'gain' can be transferred

to its new setting only when the Subjects see certain relationships that exist between the selling and the cost prices.

B. 9. Lack of Experience in the Business World

Lack of experience in affairs outside the school showed itself in four types of errors. These were : (1) In checking the value or price of articles bought, (2) In estimating the number of articles of known value that could be bought for a given amount of money, (3) In lack of appreciation of the size and relation of the denominate measures on the part of Subjects who had memorized the tables of measures, and, (4) In lack of certain, practical, work-of-the-world information.

Examples of type (1) are as follows: One Subject in problem 12 got $720.72 as the cost of the linoleum that would be needed for a kitchen floor. The writer said, " Do you think that $720 is about correct as the cost of the linoleum for a kitchen floor? " and to the question, the Subject replied, "Yes". Subject 4 in problem 7 thought that if he were buying a man's hat he would be required to pay about 25 cents for a cheap one and about 30 cents if he wanted an expensive one. Subject 18 thought he would have to pay about $45.00 for a good hat. There are also great variations in the values placed upon the 2520 pounds of wheat in problem 11. Subject 12 got $342,800.00; Subject 14, $211,680.00; Subject 17, $5880.00 and Subject 20 only $3528.00. Subject 6 of grade 1V is a very able

student. His solutions for problems of his grade were of unusual merit, yet in estimating the value of this wheat he placed a value of $10.00 upon it. When he tried to give a written solution he obtained $1312 but said he thought it should be about $2500.00 . When the $10 and the $2500 estimates were later considered together, this Subject said frankly that he did not know which amount would be more nearly correct.

An example of error type (2) is the statement that 800 hats costing $3 each could be bought for $24.00. The Subjects who made errors of this sort could be made to realize their errors if they were challenged two or three times, but, invariably the first questioning aroused no doubts. One might argue that we have here lack of attention. Probably so, but, may not the inattention be the sign of an absence of experiental ideas.[2]

In dealing with denominate measures there are many errors that show a lack of familiarity with the units talked about so glibly. Subject 12 indicated what she thought the size of a bushel measure would be. It was less than that of a gallon measure. To Subject 14 a square yard is an area about 1/36 of its natural size, and a square foot is of dimensions 3 inches by 5 inches. Subject 15 indicated what the dimensions of a 7 pound box of candy would be. The imaginary box would contain about 3/4 of a pound of 'sweets".

Subject 13 in problem ~~36~~ 9 got 53 miles as the distance the

car would go on 1 gallon. This result did not prevent him from estimating that a Ford car could go about 9 miles on a gallon of gasolene.

In the questioning on the problem, " Milk costs 14 cents a quart, what do 2 quarts cost?", it was discovered that Subject 4 knew no money coins except 'coppers'. The five and ten cent pieces were unknown. A lack of familiarity with money was responsible also for the failure of Subject 2 on the problem, " I bought a bicycle for $18.25. I spent $3.90 on repairs. I then sold it for $24.00. How much money did I make?", for she said, pointing to the $18.25, "I cannot understand all this money". This same Subject understood problem 3 which was of like nature but involved smaller sums of money.

These and other examples prove that our Subjects are being asked to solve problems concerning which they do not have complete understanding. Wide experience gives a background of information and comparative data which may be used later in problem-solving.

While lack of experience is a handicap in the solution of practical problems, a pupil may have had the past experience and then use it to his own disadvantage. Subject 12 drew upon his past experience for certain information instead of getting it from the problem in the form in which it was to be used. The problem said that 10 pages of a book were being read per hour but the Subject said that sometimes it

took over an hour to read a page and he let this truth affect his treatment of the problem under consideration.

B. 10. Confused by Large Numbers

Problem 5 read : " Find the cost of 7 pounds of candy at 35 cents a pound". After this problem was worked each Subject was asked to solve, " Find the cost of 12367 pounds of coffee at 47 cents a pound". The different methods followed by three Subjects in the two solutions are given below.

Subject	Method in Problem 5	Method in Special Problem
2	35 cents x 7	Find how many 7's there are in 12367.
12	35 x 7	Find how many 7's there are in 12367.
1	Added a column of seven 35's	Subtracted 47 from 12367.

Subjects who have a perfectly generalized manner of dealing with a given type of situation are not disturbed by the change in magnitude of the numbers involved. A given method may be associated with certain types of situations which in the experience of the child have been limited to certain limited ranges of information. If the idea of, say, 'Given the price of 1 article, find the price of any number of articles' has not been abstracted from its particular settings, the mere change of numbers may convert a familiar type of problem into an unfamiliar one.

B. 11. Errors in Analysis into Knowns and Unknowns

Nine kinds of errors were made by the various Subjects

in naming the knowns and unknowns. As these are named examples of each are given.

1. The classification was correct as far as it proceeded but it was incomplete.

Subject	Problem	Error
18	10	Said that perimeter was the only known element.
13	8	Omitted the rate of reading as one of the knowns.
10	3	Said that the only known was $3.28
16,11 and 14	11	Said that the only known was the price per bushel.

2. In a two-step problem the knowns and unknowns of only the first step were isolated.

Subject	Problem	Error
23	12	Said that he was asked to find how much linoleum was needed, whereas he was asked to find the cost.
13	4	The Subject estimated the answer to be 40 pages instead of 6 hours. The first step in the solution was attempted mentallyand the second part of the problem was disregarded.

3. Names a known as an unknown.

Subject	Problem	Error
12	5	Said that one unknown was ' How many pounds are there?'.
4	3	Said that the cost of the doll was unknown.
9	8	Named 'How many pages she reads in an hour' as an unknown.

4. Names an unknown as a known.

Subject	Problem	Error
48	10	Said that the area was one of the knowns
17, 1, 19,23	4	Gave 'How many seats are empty' as one of the knowns.

8,13	3	'How much more she needs' given as a known.
19	5	'Cost of 7 pounds' given as a known.
16	7	'The number of hats for $24' given as a known.

5. The entire written content of the problem given as known.

This happened with Subject 48 in problem 13, with Subject 9 in problem 5, and with Subject 2 in problem 5.

6. What the Subject knew or did not know was taken as the known or unknown.

Subject 22 worked problem 7 mentally. She therefore knew that the answer was 8 hats. Instead of naming the correct knowns and unknowns she gave as her classification :

Knowns: Cost of 1 hat; Cost of 8 hats.
Unknowns: Find cost of 8 hats.

7. The written solution or the oral answers reveal the fact that the Subject has proceeded to a solution without determining the knowns and unknowns.

Subject 3 gave "72+35 = 107 miles" as the answer to the problem "Jack rode 72 miles yesterday. He rode 35 miles in the forenoon. How far did he ride in the afternoon?".

Subject 4 said that the unknown in problem 7 was ' How much it cost all together?'. The unknown is 'How many hats'. The Subject has not taken the trouble to determine what the knowns and unknowns are. He merely invented or filled in the type of unknown that he had found attached frequently to the buying situation.

8. Classification was definitely wrong.

Subject	Problem	Error
9	2	The knowns given as : afternoon, forenoon, 35 and 29.
1	5	The known as 7.

9. Irrelevancy in the response.

Subject 9 said that in problem 4 the knowns were: 80 men, 94 women and the answer.

There are very great differences in the clearness with which different Subjects found the real problem and in the reasons which they gave for their methods. Illustrations from the solutions given for problem 4 will make this evident. One Subject wrote as follows: " A train has seats for 220 persons. After 80 men and 94 women have taken seats there are left 220 - 174 = 46 seats left. I first wrote that a train had 220 seats in it. Then I said that 80 men and 94 women got in it. Then I told how many seats were left. To do this I subtracted". Another Subject from the same class said :

" 220
80
94
394

I got 394. I added it because that what I could do it the best."

A third Subject gave as his work:

" 220
80
94
394

I did addit and it might be it but I am not shore of it. Or you should subtract it or multiply."

Yet another Subject said:

" 80
94
174

The reason I did the problem this way was because it was easier. The 174 seats means the amount left of seats."

When some Subjects attempt to solve problems they are not definitely conscious of the known and unknown elements. Sometimes this failure to analyze the problem leads to wrong solutions, but, frequently it does not prevent a Subject from arriving at a correct answer. Particular

problem settings call forth directly appropriate methods of treatment. Are problems so frequently given in conventional and stereotyped forms that pupils are able to recognize types at sight and are able to deal with them according to rule

B. 12. Analysis not carried to subordinate ideas.

The errors that have been assigned to this class were found in (a) the written solutions, (b) the general questioning, or (c) a special test where the selection of methods was investigated. Lack of analysis in each of these three fields was due to several causes, each of which is exemplified below.

(a) Errors in written solutions

Problems 4 and 8 were two-step problems. There were very many instances where the first step of each was taken correctly but all work stopped at that point. In problem 4 the 80 was added to the 94 and the sum named as the number of empty seats. In problem 8, the 25 pages already read were subtracted from the total 85 pages and the work proceeded no further. This same error was found in the work of six Subjects.

Subject 7 in problem 4 added 80 to 94 not because the total number of persons had to be found before the solution could be finished but because of the word 'and' between the two numbers. The problem read '80 and 94' and the wording gave the cue. Analysis was not necessary.

In problem 10 more than one pupil took 60 x 60 instead of 15 x 15 when finding the area of the square. The Subjects failed to think of the meaning of the word 'perimeter'. The word 'perimeter' stands for something that can be broken into constituent parts and the required analysis had not been made.

There were a few instances where the wrong operation was performed upon the numbers in the problem. In a simple one-step problem the two numbers were sometimes added instead of being subtracted. The Subjects read accurately but immediately forgot the relation of the two numbers to the item which was to be determined.

Carelessness in reading led one Subject in doing problem 8 to divide 85 by 10 to get 8 1/2 hours as the time required to complete the book.

(b) Errors in selection

Subject 17 read problem 7 and in advance of his solution had a vague idea of the time that it would take to read the book. When in his solution he came to 60 and 10 and these two numbers were to be divided, he multiplied them because he said he knew that it would take a long time to read the book.

Subject 14 was not critical of her methods and did not analyze the situation to make sure that the operation to be performed was the correct one. She said that if 7 pounds cost 35 cents, one pound would cost 7 times 35 cents. This error was due in part to a lack of attention.

Some errors in selection were due to lack of care in thinking about the conditions of the problem. Subject 19 in problem 8 said that it would take Alice over an hour to read a page. The concrete situation is not pictured in a Subject's mind when gross errors such as this are made. The problem stated definitely that Alice was reading 10 pages an hour. Similarly Subject 3 in problem 3 stated that to buy the doll for \$4.95 one must save \$4.95 + \$3.28. The Subject does not sense the buying situation when he makes such decisions.

(c) Errors made in the general questioning.

One interesting type of error occurred here. Two Subjects gave correct written solutions for problem 12. The method followed was that of first finding the area in square feet and then changing the square feet to square yards. In the questioning period the Subjects were asked if the problem could be worked by taking the length of the sides, changing them from feet to yards and then proceeding from this point to find the area. Both Subjects replied in the negative. The Subjects had formed a habit of analyzing problems of this type in a fixed manner and they failed to carry through an alternative analysis when the latter was suggested.

Subject 9 was very sensitive to suggestion. Without analyzing the proposition submitted, a wrong method would be accepted readily. Regarding the problem, " If a quart of milk costs 14 cents, find the cost of 1 pint", the Sub-

ject was asked if he would add 2 to 14 because there were 2 pints in 1 quart. He said, "Yes" and proceeded with the suggested addition.

In a few instances laziness appeared to be the only reason for not evaluating ideas before accepting them, Subject 9 was asked in problem 8 how long it took Alice to read one page and he replied, "Sixty". The only reason for replying with sixty was the fact that at that moment the solution had progressed to the point where he had subtracted 25 for 85 and 60 seemed to be the only number that was conveniently at hand.

Findings with respect to the forcing of analysis

When a Subject has not analyzed a problem carefully there are various methods of forcing the analysis. Examples of this that were found in this investigation are illustrated below.

(1) Direct Questioning. Direct questioning is probably the most common method of getting the desired analysis. It is so common that examples need not be given.

(2) Re-reading the Problem. Under the heading 'Errors in Reading' illustrations were given to show the effect upon analysis of having the Subject re-read the problem.

(3) By Having the Subject Explain the Reference of all Numbers that are used.

One Subject in problem 10 wrote 60 x 60 = 3600 sq.ft. The writer asked him to explain what each number represented. As soon as he had to label his numbers, he said, " I shouldn't have two 60's". The defining of the 60 drew

to his attention the fact that it was something different from what he had called it.

(4) By Use of Diagrams. Subject 14 was the only one serving in this investigation who used diagrams and graphs as aids in problem-solving. This Subject was found to be very deficient in knowledge of the number relations. Correct answers were deduced often in spite of the fact that relations were not seen, because of the aid given by the diagrams.

Subject 31 was having difficulty with the problem," Divide $125 between two boys giving one boy half as much again as the other". He gave as his solution:

" How much does each get
1st boy 125 ÷ 2 = 62 1/2 + 1/2 of 62 1/2 = 31 1/2 +
62 1/2 = 93 3/4
2nd boy 1/2 of 62 1/2 = 31 1/4
Therefore the first gets 93 3/4 2nd = 31 1/4.

The following questioning took place:

Q.: Draw a line to show how high the pile of money would be if you gave the first boy his share in 25 cent pieces.
A.: A line was drawn as follows: __________
Q.: If the other boy gets half as much again, draw a line to represent his pile.
A.: A line was drawn as follows: _______________

Without further questioning, the following solution was written :

2/5 of $125 = $50 3/5 of $125 = $75.

It is evident that the Subject gained from the questioning and the use of the graph a general notion of the relation between the two unknowns of the problem, for, without further assistance he proceeded as indicated with the

following problem: " Divide $35 between two boys giving one 1/4 less than the other".

```
Solution: " ____________________          3/7 of $35 = $15  "
            ______________                4/7 of $35 = $20.
```

(4) Imaging the concrete setting of the problem.

Subject 24 was given the problem," At 6 cents a pound find the value of 864 bags of coffee each weighing 130 pounds." The first two written solutions offered were :

```
(a)    1 lb  864          (b)     864
             130                + 130
           -----                -----
           26920                  994
             864                x   6
           -----                -----
         $355.60                $3984
```

Then came the following discussion:

Q.: What did you see when you were working the problem?
A.: The figures (meaning the digits).
(The Subject was asked to close her eyes while the problem was read aloud to her.)
Q.: What did you see?
A.: Nothing.
(The Subject was asked to close her eyes again, to listen to the story told her and to try to see all the story in picture form. This simple story was told to her. " The little girl standing in the door has a pretty red rose in her hand".)
Q.: What did you see that time ?
A.: A girl in the door. She has a red rose.
(The Subject was practised on two or three similar stories in an attempt to get her in the habit of imaging the concrete situations of the problem. The original problem was then read to the Subject and she was asked to state what she saw when her eyes were closed.)
Q.: What did you see in the picture?
A.: A whole room full of sacks. Each bag was about as big as a sack of potatoes. A little paper bag full about this high.

Written attempt number three followed at once:

```
(c)   130             864
        6           $7.80
    -----          ------
    $7.80          65120
                   6048
                  -------
                  $125600
```

There is a mistake in multiplication but the situation and content were correctly analyzed in the attempt to image the problem setting.

(5) Analysis is forced by imaging the units of measurement used in the problem.

To the question, " If wheat is worth $X a bushel, how much money should I get for Y pounds of it?", Subject 18 replied " Divide X by Y and then multiply by -----". The Subject was asked to think of the size of a bushel of wheat and to indicate what he thought it would be. His imaginary bushel was cylindrical in shape, 1 foot in diameter and 3 feet high. He was asked to indicate the volume of 1 pound of wheat. The estimated cylinder was about 6 inches in diameter and 5 inches high. Following this discussion, the Subject said, " The way to do it is to divide Y by 60 and then multiply by X".

B. 13. Failure to Establish Relationships.

Failure to establish relationships is not to be considered as a simple cause of error nor as a cause which we have not touched upon earlier in this report. If a pupil were efficient in establishing relationships he would be a very efficient problem-solver if he could but identify the operation needed to fit each problem. Under the present heading are grouped many deficiencies in which appear weaknesses in argument and deduction when the Subject is in possession of the specific data required. No attempt is made to arrange the errors in the order of their importance.

(1) The inability when solving one-step problems to see the relationships that would appear if in the equation form $X + Y = Z$, or $X = Z - Y$.

Subject 1 was asked " If you knew how many seats were on a train and you knew how many seats were taken, how would you find how many seats were empty? ". The reply was, " Subtract the number empty from the number full". The relationship here is that expressed by the equation $X = Z - Y$ but the Subject expressed it as $X - Y = Z$. Similarly Subject 2 in problem 3 said that Alice needed to save more than $4.95. She gave the relationship as $X + (\ ) = Z$ when it should have been $X - Y = Z$.

(2) The inability when solving one-step problems to see the relationships as they would appear if in the equation $XY = Z$ or $X = Z/Y$.

To find the price of 7 pounds of candy at 35 cents a pound, Subject 1 said that you should divide 35 by 7 to get the answer. The correct form would be $XY=Z$ but the Subject gave $X/Y = Z$. Subject 4 said that 7 pounds cost 35 cents and that therefore 1 pound would cost 35 x 7. The Subject is wrong here in the primary assumptions but the argument shows the same error as does the first one mentioned.

(3) The same inability as named in number (2) but in this instance the one number is derived from some other unit and although the Subject has the necessary information, this extra item must be held in memory.

In reply to the question, "If you know the cost of 1 quart of milk, how do you find the cost of 1 pint? " Subject 4 said " Add the cost of 1 quart to the cost of 1 pint". As in number (2) above the true relationship is

expressed by X = Y/2 and the pupil gave it as X + Y/2 = ( ). Here the number 2 must be held in mind after the necessary relation of the pint to the quart has been established.

(4) Where the relationship is expressed by X = Y / Z.

Subject 1 was asked how many \$3 hats could be bought for \$24 and he replied 24 times 3. The relationship was seen as X = Y x Z.

(5) Relationships existing between the subtrahend and the minuend(negative values not considered).

Subject 4 in problem 1 said " Take the most away from the other one" when asked how to find how much one is older than the other when the ages of two boys are given. Instead of X - Y = Z he gave Y - X = Z.

(6) In two-step problems the relationships in one part may be seen though those in the other part remain undiscovered.

Problem 4 involves the relationship, X + Y = Z (80+ 94 =Z) and 220 - Z = X. Subject 2 in solving this problem proceeded as follows: 94 - 80 = X. The Subject has somehow sensed the correct operation to be performed in the second step but he does not see the relationships that are determined by the first equation.

Subject 4 makes the same general type of error when he proceeds thus: 220 + 80 + 94 = X. Here the first step is correct and the second is wrong, as the work is equivalent to 80 + 94 = Z ; 220 + z = X.

(7) The application of the rule of three in proportion.

The correct form is here X : Y :: Z : ( ). Problem 11 gives the value of a bushel of wheat to be $1.40 and asks for the value of 2520 pounds. The solution is of course 60 : $1.40 :: 2520 : X. Subject 20 gave the following:

" 60 pounds is worth $1.40
2520 pounds is worth 2520 x $1.40 = $3528.00 "

Differing from this example only in the generality of the statements is the problem " If you know how far a car goes on 8 gallons of gasolene, how can you find how far it will go on 3 gallons? ". The Y is here given a specific value but the principle remains the same. Subject 19 gave the following argument:

" 8 gallons takes him 160 miles
1 gallon takes him 160 ÷ 8 or 20 miles
3 gallons will take him 160 ÷ 3 or 53 1/3 miles."

(8) Two unknowns determine the relationship between X and Y but one of them is overlooked.

In " How do you change pounds of grain into bushels of grain?" the unknowns are (a) What operation is to be performed?, and (b) How many pounds are there in a bushel of this particular kind of grain? Subject 49 said, " Divide by 60" thereby showing that the unknown (b) had been overlooked.

(9) Particular business applications of (1) of this list.

$X \overset{-}{\cancel{+}} Y = Z$ becomes Cost Price - Loss = Selling Price in certain situations. In the problem " A boy sold his bicycle for $32. He lost 1/9 of what he paid for it. What did it cost him?", the situation is complicated because of

the presence of a fraction. It is worth noting that Subject 14 instead of seeing that Cost Price - 1/9 Cost Price = Selling Price, gave as her solution the following:

```
" 9) 32              32     "
      3 5/9           3 5/9
                     35 5/9.
```

(10) Inability to deduce the relationship between two units of square measure when the relationship between the corresponding linear units is known.

Subject 14 knew how many feet there are in 1 yard but he did not know how to deduce from that the relation of a square foot to a square yard.

(11) An equivalent correct method is refused because it is not seen to be equivalent to the particular one used by the Subject.

To find in square yards the area of a rectangle 12 feet long and 9 feet wide, one may proceed by either of the following methods:

(a) 12x9=108; 108÷9=12. There are 12 square yards.
(b) 12÷3=4; 9÷3=3; 4x3=12. There are 12 square yards.

Subject 14 used method (a) but in the discussion that followed he denied that method (b) would be satisfactory.

(12) The table of denominate measures is memorized but the actual relation between the measures is not known.

In response to the question, " Should you divide by 60 or multiply by 60 to change pounds of wheat into bushels of wheat? " Subject 17 said, " Multiply by 60" showing inability to apply the denominate number fact that was given.

(13) Does not see the relation between certain parts of geometrical figures.

Subject 11 who gave 30 x 30 as the answer when finding the area of a square of which the perimeter was ~~30~~ 60 feet, did not know the relation that the perimeter bore to the length of one side.(Neither did he know the relation of the unit of area to the linear unit.)

It was stated earlier in this report that the presence of fractions complicates the problem of seeing relationships that exist. The following examples illustrate this fact:

Subject 16 had great difficulty in seeing the relationship that existed between 8/9 and 9/9. As he was worrying over the problem he was asked to think aloud. His monologue ran as follows: " He lost 1/9. The $32 is 8/9. But I can't find 1/9 of 32. If I could find 1/9 of the cost I would multiply by 9 and get the answer. Eight goes into thirty-two four times. That would be 1/9 of it. The $32 is 8/9 of it. If I divide 8 into it well ---- that shows me how many ninths there were. No. It shows me how much 1/9 of it is. The cost of the bike was $36.00".

Subject 14 after drawing a line as ______________ to represent the cost price, and after cutting 1/9 from it to represent the loss, was yet unable to identify the $32 with 8/9 of the line, although she realized that the selling price was 8/9 of the cost price. Two facts were held in isolation, (a) the selling price was $32, and, (b) the selling price was 8/9 of the cost price.

B. 14. Lack of Imagery

We have already seen how the imaging of the concrete situation aids in the analysis of a problem. There are certain instances where the errors would have been avoided had this imagery been present to aid the Subject. Subject 12 in problem 7 chose as the required method " The number of hats would be 3 times 24" after giving a written solution where she got $72 as the required number of hats. The Subject was asked to close here eyes and think of the problem that would be in the mind of the lady who went to purchase the hats. When this was done she said that she would get 8 hats. Subject 22 estimated that at $1.40 a bushel, a load of wheat was worth $5.20. Subject 14 estimated that in problem 12 the linoleum for the floor would cost only $3.00, showing that he failed to image either the floor or the unit area of one square yard.

B. 15. Inaccurate Graphical Aids lead to Errors.

This error occurred because the Subject was required to use a graph. He was not perturbed by the meaning of a straight line graph but the analysis that should accompany its use had not been made. Subject 15 attempted to use a grph line that would illustrate " 1/4 of the work was done in 3 days". The supposed quarters of the line were far from being equal in length and this led the Subject into serious trouble later when he returned and based certain

calculations upon the measured length of one of the unequal segments.

Few Subjects used graphical aids to assist them in their analysis of a problem. In the work of 52 Subjects as given for over three hundred problems, there were but eight single problems for which any graphs or diagrams were drawn and these eight different diagrams were used by seven different Subjects. No single Subject had formed the habit of using graphical analysis before attempting to solve a problem. The relation of the ability to draw simple line graphs and the ability to solve problems is such that pupils who are most expert in the former are also most expert in the latter operation.

When Subjects are forced to use a graph in analyzing a problem great individual differences are seen in the degrees to which the lines drawn actually represent the magnitudes in the problem. Relative to problem 4 all Subjects were instructed as follows: " Draw a line to show how many seats there are on the train. Draw one to show how many men got on the train. Draw one to show how many women got on the train. Draw one to show how many seats were left empty". The responses of five Subjects gave lines that were of the following proportionate lengths:

Subject	Line 1	Line 2	Line 3	Line 4	Average per cent Error
23	10	4.0	5.6	2.0	12
22	10	5.6	7.3	3.3	56
20	10	7.3	8.4	3.0	68
8	10	9.3	5.4	3.3	72
9	10	10.0	10.0	10.0	219

The true ratio was 10 : 4 : 4.3 : 2.1

Subject 16 solved problem 2 correctly but drew a graph line for Dick's age that was longer than the line representing John's age.

Subject 20 gave a correct solution for problem 4 and then drew the four lines in the relative lengths 36: 25: 30: 12.

Subject 17 gave a perfect solution for problem 8 and then in the graph stated that the portion of the book read was about 35% greater than the 60 pages that yet remained to be read. In problem 7 this same Subject drew a line 26 units long to represent the cost of 8 hats and then a line 16 units long to represent the cost of 1 hat.

These examples show once more that there are various ways of arriving at an analysis of a problem. These Subjects have other ways of seeing that certain relationships are involved. Their failure was not due to the fact that they did not know what was demanded in the graph.

As pupils get older and progress through the grades, although~~t~~ they do not actually use graphical aids, they increase in ability to use them upon demand. This is due to growth in ability to analyze problems. Of the 23 Subjects who attempted problem 4, twelve got a correct solution on the first attempt, while eleven obtained incorrect answers. The percentage error in length of the lines drawn in all the graphs was for those who got the answer correct 45.6, and for those who got wrong answers it was 108.8. The average deviation for the first group was 29.3

and for the latter 53.6%. 83% of the first group had lines 2, 3 and 4 of the correct order of magnitude but only 55% of the latter group succeeded to this same extent. There is a correspondence here between success in problem- solving and the correct use of graphs. Is it that those who can solve problems happen to be those who understand the use of graphs or is it the case that the real parallelism is between problem-solving ability and degree of analysis, which merely happens to be measured here by the particular method under discussion?

C. 16. Incomplete Notions Concerning the Operations

The data relating to this topic will be presented separately for each of the four fundamental operations and will relate in order to addition, subtraction, multiplication and division.

Incomplete Notions Concerning Addition

Several Subjects were asked when they would add two numbers that they happened to find in a problem. The answers revealed whether they knew the meaning of the operation of addition. Subjects 15, 16 and 23 answered correctly with replies such as the following:" To find the total". Subjects 18 and 20 said that they would add when they wanted to find the cost of two or more things. Here one special sort of situation was singled out as demanding the addition operation. A few Subjects seemed to be unprepared for the business of selecting the situations and merely stated that they added when they were given adding questions, while Subjects 2 and 7 said that they added in

problems when they wanted to get the answers. Subject 3 had replaced the main notion inherent in addition by an associated one. He had been told repeatedly to add only numbers that were of the same denomination or unit so he said he would add two numbers in a problem if they were both of the same sort. Subject 13 said he would add the numbers when he wanted to multiply them, which proved to mean that he had in mind the explanation of his teacher that multiplication is but a new way of adding.

The following examples show that the meaning is understood but that the application is not identified.

Subject 3, Problem 3 :

Q.: What does subtraction mean?
A.: Take away.
Q.: In this problem what are you trying to find?
A.: How much more she needs.
Q.: How do you find it?
A.: Put down the numbers.
Q.: Yes, but should you multiply, subtract, add or divide?
A.: I don't know.

Subject 2, Problem 3:

(The Subject tried to add something to $3.28 to get $4.95.)

Q.: What are you doing here?
A.: Adding up.
Q.: How can you find how much she needs?
A.: She must save money.
Q.: How will you find how much?
A.: By adding more money to $3.28.
Q.: If one boy has $1 and another boy has 1 cent, how much money has one boy more than the other?
A.: 99 cents.
Q.: How did you get that?
A.: Subtract.

Q.: If she has this much money ———————
and she needs this much ————————————
how much must she save ?
A.: Add enough on to make the other.
(This Subject adds or subtracts according to the wording of the problem. The general principle has not been grasped.)

Subject 30, Problem 6

Q.: How did you get this answer of 7 cents ?
A.: The pints make a quart. 7 and 7 are 14.
Q.: If a quart costs $76 what does a pint cost ?
A.: 38.
Q.: How did you get that ?
A.: 38 and 38 make 76.
Q.: How did you get the 38 ?
A.: I got the number by trying that would add to itself and give the 76.
Q.: If three pieces of cloth cost $84, what will one piece cost ?
Subject now tried 20+20+20; 25+25+25; 30+30+30.

The examples here given suffice to show that there is great variation in the general working knowledge that pupils have of the application of the fundamental process of addition. Many have not isolated the essential feature of all adding situations and are therefore inadequately equipped to identify the new situations that arise. This handicap may not reveal itself in the familiarly-worded problems but if a problem involving new units is presented, these pupils are not able to extract the element that is identical in the similarly-constructed problems.

Each Subject was asked to manufacture a problem in which addition was demanded. Not general features of situations but specific applications had to be isolated. Subject 6 gave this example: " Jean got a coat for $25 and a hat for $5. How much did she spend ?", showing a

clear understanding of the situation and of its treatment. The same understanding accompanied by less facility in the use of English prompted **the problem** , " If Mary is 10 years old and John is 5 years old, how many years are both together ?". Subject 7 was ready to deal with the problem when it arose but in his mind the idea of a series was uppermost and the idea of summation took second place. He offered as his problem: " I bought 25 cents worth of candy, 75 cents worth of pop-corn and $1 for a pair of stockings". An answer-seeking pupil said, " I bought a stick of candy for 6 cents, a dozen pencils for 15 cents and a bottle of ink for 15 cents, find the answer". Subjects 1 and 2 followed the line of least resistance and constructed problems like those previously used on the same day and they therefore gave as an addition problem one similar to ' If Bill had $5 and Jack had $11, how much more had Jack than Bill ?. Subject 5 knew little about the use of addition. His problem was " If Jack went to the store for 25 cents worth of oranges and Mary went for 12 cents worth of oranges and oranges are worth 10 cents a piece, how many more would Mary get than Jack? ". This Subject said that he did not know when he would add two numbers that he found in a problem. As a matter of fact when he later met the adding situation in problem 2, he subtracted 29 from 35 to get his answer.

All the Subjects assisting in this investigation were able to add simple examples at least. The fact that they

could perform the operation itself was no guarantee that they understood the process. We have seen that the understanding of the process is complete in some instances and quite inadequate in others. We have no information as to when such understanding should be given but it must be acquired sometime. Other sections of this report show that in the various grades those pupils who have the most complete understanding of the fundamental operations are those most capable of solving various problems.

Incomplete Notions Concerning Subtraction

To the question, " How do you find the difference between any two numbers " the following answers were received:

Subjects 6 and 9 : Subtract
Subject 7 : Add
Subject 8 : By adding or subtracting
Subject 2 : You can tell if one is bigger than the other when you look at them
Subject 1 : How many times one is bigger than the other.

It is clear that these ideas include in their range addition, subtraction and division.

To the question," For what reasons would you subtract two numbers that you found in a problem?" responses were given that show a variety of ideas. These responses are summarized below.

Central Idea	Subject	Response
To find the remainder	17	When one is larger than another to find how much it is larger.
	16	When you want to find the remainder
	15	To get what is left when the smallest is taken from the largest
	11	When you want to find what is left when you take one from another

Central Idea	Subject	Response
Loss or gain as a special case of difference	22	To find the loss or the gain
	23	To find gain or loss
It is a subtraction or take-away question	13	When you have a subtraction question
	9	In a take-away question
A result is confused with an operation	8	When you are trying to find something that is less
	20	When one costs more than another and you are trying to find what the lesser one costs.
Size rather than relationships	3	When one is higher than another
Adding backwards	2	When you want to add down from one to the other
Confusion	12	When you want to find a lot of numbers
Partition	18	To find the half

In attempting a solution for problem 4, one Subject proceeded only as far as 80 + 94 = 174. In the oral discussion following this attempt the writer gave this example, " A train has seats for 100 persons. After 1 man and 94 women have taken seats, how many seats are empty? ". The Subject replied promptly, "Four". For the original problem he now says, " See how many you can add on to 174 to get 220". This he tried to do without success. This Subjects allows the size of the numbers and the special setting of the problem determine his choice of the operation to be used. It should be noted that the Subject had no difficulty in subtracting 174 from 220 when asked to do so.

This Subject once more revealed his lack of information about subtraction. The writer said, " If you know how far it is from one place to each of two towns down the same road, how may you find how far apart the two towns are?", and illustrated the problem with three lines placed as here shown. The Subject said, " Divide or multiply".

In response to the direction " Make a problem with two numbers in it where you would subtract to get the answer" the ideas summarized below were included.

Central Idea	Subject	Response
To find a remainder	4	If Ruth said she was 10 years old and she is only 6 years old, how many do you have to take away to get her correct age?
	1	If John had $9 and he gave Dick $3, how much had John left?
Essential idea is present but vague because of irrelevant ideas or because of poor English	9	A man had 12 pigs; another had 5. How many was there between one from another?
	2	John has a little sister. She is 10 years old. John is 14 years old. There are three of them in the family. The little one is 4 years old. How much younger is the little one than John?
Confusion	24	I went to the store and bought a hat for $1.50. I bought a lamp for $3 and a coat for $5. Find the answer.
A number fact not a problem situation	8	Subtract 10 from 11.

The above data show that for subtraction as for addition a great variety of associated ideas is entertained. From definiteness, through indefiniteness, to irrelevancy, are ranged the responses with respect to their relation to the idea of subtraction. Some Subjects gave wrong answers as to the purpose of the operation or when searching for an application, and yet they were able, when confronted by some problem involving subtraction, to give correct solutions. The question arises concerning the extent to which and the time when pupils should be practised in identifying applications of the principle of subtraction.

Incomplete Notions Concerning Multiplication

The following questions were asked the Subjects: (A) When would you multiply two numbers that you found in a problem,? (B) What problem can you give me with two numbers in it where you multiply to get the answer? (C) How do you find the cost of any number of pounds when you know the cost of one pound? (D) If you knew the cost of one hat, how could you find the cost of several hats? and (E) If you know how far a car goes on one gallon of gasolene how can you find how far it will go on any number of gallons?

The replies listed below show certain characteristics of the ideas held by the several Subjects.

Subject	Question	Reply
6	A	Same as when you add but it's shorter
	B	Jean got a coat for $20, a hat for $20 and a pair of shoes for $20.
	C	Multiply the number of pounds by the cost of one pound.
21	A	To find the product
	C	Multiply the cost of one pound by the number of pounds
	D	Multiply the cost of one hat by the number of hats
	E	Multiply how far it goes on one gallon by the number of gallons.
9	A	You never do it
	B	A man had 4 horses. Another had 54. Get 4 times more then 54.
	C	Divide
	D	Add
5	A	No response
	B	If Mary had 2 oranges, if Jack had 12 oranges and he wanted to give each boy one, how many would each boy get?
	C	Add what one pound costs
	D	Divide
4	A	No response
	B	A girl is 16 years old and one is 15. How much older is she?
	C	Add it
	D	Add

No two of these questions were given to the same Subject on the same day. Whatever uniformity appears in the various replies must be due to habitual modes of thinking about multiplication. Subject 21 has very definite and exact notions concerning the operations to be performed. In question C Subject 6 identifies the application when it is given , but, in B he has difficulty in himself finding an application, partly because his idea of the purpose of the operation, as revealed in his reply to (A), is that it is very much the same as addition. Subject 9 evidently does not understand what

[illegible]

[illegible]

multiplication means; it is 'times more than', 'division' and 'addition' in turn. Nevertheless when on another occasion this Subject met the problem " Find the cost of 7 pounds of candy at 35 cents a pound" he said that 7 pounds would cost 7 x 35 = 245 cents. This pupil has learned to do the correct thing in the multiplying situation although he has little idea concerning the meaning of the thing he is doing. Subject 4 talks in terms of addition and subtraction, mostly addition, and true to his method of thinking he attempts to solve problem 5 as follows: " 35 + 7 = 42".

The central ideas expressed in the various replies were as follows:

Question	Idea	Subjects
A	No response	1,3,4,5,7,19 and 20.
	To find product	21
	To find cost	44,22 and 14.
	To find a difference in cost	23
	To get more	15,16.
	Division	2
	Addition	6

(Subject 15 has abstracted from the operations the ideas of making more and of making less. He has begun to classify these experiences but he has not yet noted the differences between addition and multiplication. He is generalizing but his present generalization must be subjected to further revision.)

B	Addition	6
	Subtraction	4
	Division	2
	A form of operation carrying sign	1,8 and 9
	Confusion	5

C	Multiplication	18,17,16,15,13,10,11,6,19, 20,21,22,23,14.
	Division	9
	Addition	1,2,3,4 and 5.
D	Multiplication	18,17,16,15,10,6,19,20,21, 22 and 23.
	Division	5,8 and 12.
	Addition	1,2,3,4,7 and 9.

(Subject 5 thinks of addition at one time and of division at another; Subject 7 of multiplication in one situation and of addition in another.)

The grade 111 Subjects all talked of addition and when they tried to solve problem 5 each of them tried some sort of addition to get the solution. All pupils above grade 111 worked problem 5 correctly as far as method was concerned. Whatever the limitations of knowledge with respect to the meaning of multiplication, the Subjects above grade 111 have developed certain working habits for problems worded in familiar ways. The 'Find the cost of ---- at so much a pound' type gives little difficulty. On the other hand great difficulty was experienced in problem 6 after the distance travelled on one gallon had been found.

Problem 6 was not given below grade V but two Subjects of that grade and one Subject of grade V11 who experienced no difficulty with the 'Find the cost of --- at so much a pound' type failed to find how far a car would go on 3 gallons of gasolene when they had already stated that 1 gallon would take it 20 miles.

One notes that it is not always easy to turn from theory to practice or vice versa. In the general discussion about the problems, Subjects 11 and 13 each said that to find how far a car would go on any number of

gallons you should multiply how far it goes on 1 gallon by the number of gallons, and yet when they faced the problem of applying this fact, they failed to apply those principles that were so readily at their command in language form.

Two weaknesses appear in the above summary, (1) The inability to identify the application, and (2) A lack of understanding of the real significance of the process. The following conversation illustrates the difficulty that one Subject had in identifying the application of one fundamental process used in problem 12.

Q.: If you have 108 square feet and you thought that there were 5 1/2 square feet in one square yard (this had been stated by the Subject) how would you find the number of square yards ?
A.: Multiply.
Q.: Which is larger, 1 square yard or 1 square foot ?
A.: One square yard.
Q.: How would you change square feet to square yards ?
A.: Multiply.
Q.: Would there be more square yards or more square feet ?
A.: More square feet.
Q.: How then would you change square feet to square yards ?
A.: Multiply.

Subject 8 in problem 5 showed a lack of understanding of the meaning of the process rather than a lack of facility in identifying the application. He began with the assumption that 7 pounds cost 35 cents. He multiplied to find the cost of 1 pound. Later when he was led to see that 1 pound cost 35 cents, he divided by 7 to find the cost of 7 pounds. Evidently neither multiplication nor division was understood. When by questioning

he was made to appreciate the real problem he said to add 35's to find the answer. At one stage of the questioning the following conversation took place :

Q.: If you know the cost of 1 pound, how do you find the cost of 5 pounds ?
A.: Add them up.
Q.: What is 5 times 6 ?
A.: 30
Q.: What does 5 times 6 mean ?
A.: I don't know.
(This pupil has memorized certain tables without knowing much about their significance.)

Incomplete Notions Concerning Division

A review of the pupil's work in division and of the answers given during the questioning is not a simple operation. Apparently one cannot teach division as a method of manipulating numbers and be sure that when the pupils have learned how to do the mechanical work they will understand division. There appears to be a gradual development of the concepts of the meaning of each fundamental operation. The illustrations given below have been selected to show that there is growth in completeness of definitions, principles and generalizations. References throughout are to the follwoing questions and directions: (A) When do you divide two numbers that you find in a problem ?(B) Make a problem with two numbers in it which you divide to get the answer, (C) If you know the cost of several pounds, how do you find the cost of one pound?(D) If you know how much one hat costs and you know how much money you have to spend, how can you

find how many hats your money will buy?and (E) Solve the problem " When 1 hat costs $3 how many can be bought for $24 "?

Subject 6 said:

(A) When you know the cost and you want to find 1/2 or 1/4 of it.
(C) Divide cost of several pounds by how many pounds there are.
(D) Divide the amount of money by the cost of 1 hat.

(E) $3\overline{)\,24}$
8 hats.

The responses here are quite satisfactory particularly when one considers that the Subject was but nine years of age and that he was just beginning the work of grade 1V. He thinks of division in terms of simple examples but he understands the operation quite well.

Subject 15 answered :

(A) To get less
(C) Divide
(D) Divide cost of 1 hat into the amount of money you have.
(E) $3\overline{)\,24}$
8

These replies are correct except for the error as to the purpose of division. The Subject has generalized from the cases that he has worked and in these division has meant 'to make less'. Subject 15 with a little assistance would probably soon have correct notions of this process.

Subject 23 said:

(A) To find the cost of something that doesn't cost so much.
(C) Divide the cost of several pounds by---- (incomplete)
(D) Divide the cost of 1 hat by the amount of money you are going to spend.
(E) " $3 buys 1 hat. $24 will buy $\frac{24 \text{ x } 1}{3}$ = 8 hats."

Here the practical work is correct but the principle of operation has not been clearly formulated. There are no gross errors but (A), (C) and (D) are imperfect.

Subject 2 replied:

(A) When you are going to divide anything in two
(B) Mary has an apple. She has to divide it with her little brother
(C) Add down that much until you get to one pound
(D) Adding up what you pay for the hats
(E) 3
3
8 hats

This Subject wrote 3 + 3 + 3 + + + until she got a total of 24. She then recorded 8 as the number of items that were added.

The guiding ideas in this work are correct but the theory is all inadequately formulated and the method of manipulation has not been developed.

Of a different nature were the responses of Subject 4 , for they reflect confusion of thought and ignorance concerning the meaning of the division process. In an earlier part of this report it was shown that subtraction was the only process that this Subject understood. To the present questions his answers were as follows:

(A) -----
(B) -----
(C) Subtract it
(D) Add
(E) " 24 x 3 = 72".

How understanding of theory and ability in manipulation may stand in marked contrast is illustrated by the work of Subject 10. To (D) he replied, " Take away cost of 1 hat and find the money left", but, for (E) he gave ," L hat costs \$3, For \$24 you can buy 3)24
8 hats can be bought."

This survey of the understanding that our Subjects had of the four fundamental operations indicates that the pupils had been subjected to widely varying experiences. The different definitions, principles and generalizations are not mastered at definite age-levels nor when pupils have reached fixed stages in mental maturity. The whole scheme of acquisition is rather haphazard. The pupils are alike in the one respect that they can do the mechanical work of any of the four fundamental processes. If any other phase of the total situation be isolated it is found that there is a marked irregularity in the time of its mastery. There may have been too little definitely-directed teaching of all the supplementary associated notions that make up enriched concepts of the four operations. It looks as if the pupils began with some vague notions as to the meaning of all their number manipulation and gradually as they met situation after situation calling for the application of one of the four processes, they slowly and probably mostly unaided clarified their notions as to what all the manipulation had been about. Some of the above evidence shows that what a child actually does in a given situation is directed not by consciously-formulated knowledge of what should be done, but as the result of recognition of familiar aspects of situations calling for specific methods of manipulation. The thinking of Subject 9 about multiplication gave an example of this type of behavior for when he solved problem 5 he wrote out a very correct solution.

C. 17. Guessing

The work of many Subjects convinces us that pupils resort to random trials when the course which they thought to be correct is found to be wrong. These pupils do not evaluate and make comparisons. Problem-solving is for them the occasion for applying a known method and it is not conceived as a process of analysis, selection, application and evaluation. The occasions on which Subjects were observed to be guessing were:

1. During the first attempt at a written solution.
2. When the questions asked were not understood because of language or symbol difficulties
3. When questions were not analyzed so that relations could be seen.

Guessing during the course of the first written solution

In problem 19, Subject 17 wrote:

$9\overline{)\,60}$ $9\overline{)\,64}$ $9\overline{)\,35}$ $9\overline{)\,60}$

Q.: What are you trying to find?
A.: (Pointing to the 32) A number bigger than this one that 9 will go into.
(The discussion revealed the fact that the pupil was trying to begin with some number larger than 32 into which 9 would divide. He was intending to proceed by trial, taking 1/9 from different numbers until the correct one was found.)

In the same problem Subject 18 wrote: " 36 ÷9 = 4. He paid $36 for the bicycle."

Q.: From where did the $36 come ?
A.: Nine wouldn't go into 32 equally so I took the next number that nine would go into equally.

When the pupil arrives at a correct answer by such means as this it is quite impossible for the teacher, under ordinary conditions, to know what the reasoning was or to know that it was not correct at each step along the route.

Other examples of first solutions that were marked by guessing were:

Subject	Problem	Procedure
3	7	3 x 24, then 3 + 24, followed by 3 ÷ 24 = 8. The answer was discovered before a written solution was attempted. The Subject's original estimate of the answer was correct. In the solution he was searching among methods for one that would give the correct answer.
11	9	" $\begin{array}{l}3\underline{)8\;}\\ \;\;2\end{array}$ $\begin{array}{l}3\underline{)160\;}\\ \;\;\;53\end{array}$ $\begin{array}{l}8\underline{)\;160\;}\\ \;\;\;\;\;20\end{array}$ " .
15	19	" $9\underline{)\;32\;}$ " Q.: Why do you divide 32 by 9? A.: I didn't know what to do so I took a wild guess at it.
14	19	" I added 9 to 32 ". Why " I was just guessing".

Guessing because question was not understood

Subject	Procedure
9	Q.: What is the cost of Y pounds of candy at Z cents a pound? A.: Subtract.
4	Exactly the same answer was given as in the last.

(Both Subjects were here confused by the use of the numbers X and Y.)

Guessing because question was not analyzed so that relations could be seen.

When Subject 7 was asked " If you know how many pages one reads in an hour and you know how many pages there are in a book, how long will it take to read the book?", he answered"add". The guess in this instance was directed somewhat by the wording of the problem ' You know how many pages ---- and you know how many pages ---- ' which introduced a large element of repetition.

Pupils sometimes resort to guessing. The more capable pupils guess only when they lack data for analysis. The seven Subjects against whom are marked the largest number of guesses were the seven Subjects who were lowest in attainment in most phases of problem-solving ability. Subject 6

guessed in one novel situation and yet when he faced another very difficult problem his behavior was of a markedly different sort. He was given problem 13 but having recently left grade 111 he was not ready for that type of problem. Nevertheless when the various formulae were presented to him he chose the one involving r^3 because he said he was asked to find a volume and r^3 rather than r^2 or r would be likely to give a volume. The Subject was prevented from guessing here because of certain information which related to the selection in hand.

C. 18 Confused by Efforts to find a Familiar Form of Solution.

In the problem " A man can do 1/4 of a piece of work in 3 days, how long will it take him to do 1/20 of it?" pupil 32 wrote as follows:

3 days he does 1/4 of his work
Therefore 1/20 of his work takes 1/4 of 3 days = 1/4 of
1/5 + 1/20 of 12 days = 12/20 = 6/10= 3/5.
1/20 of his work is done in 1/20 of all the time
= 1/20 ÷ 12 = 1/20 x 1/12 = 1/240 x 1/12 = ---

Seeing that the Subject was in trouble the writer interrupted the work and directed the following conversation:

Q.: What seems to be the matter, Bill ?
A.: I can do it but I can't write a solution.
Q.: Will you tell me how you think through the solution ? Don't bother about writing it out.
A.: It took me 3 days to do 1/4 of my work. It would take 1/5 of that time to do 1/20 of the work. The complete time takes me 1/5 of 3 days = 3/5 of a day.

Examples such as this show the uselessness of giving type forms of solutions to pupils who have not had practice

in formulating their own solutions. Should pupils be allowed to devise their own forms of statements for solutions until such time as they are very familiar with the method of solution Should standard forms be given as economical modes of procedure after the child has had wide experience with the type of problem considered Evidently pupils are handicapped when they must try to fit their mode of thinking about a problem to the stereotyped written forms which they have been taught.

The written work of Subject 23 when solving problem 9 shows that he had considerable trouble in getting the required standard written form for his problem in proportion. He wrote as follows:

It takes -----
160 miles it uses 8 gallons -------
It uses 8 gallons of gasolene going 160 miles
It uses 3 gallons of gasolene going $\frac{3 \times 160}{8} =$ 60 miles.

C. 19. A Stereotyped Form of Solution is Applied Though not Fully Understood.

A Subject may use a stereotyped form of solution and arrive at a correct result or in trying to use it he may fail to get the correct turns and twists at each step. The examples cited examplify both types of deficiencies.

Subject 15, Problem 9 :

Q.: A car goes X miles on a gallon of gasolene. How far will it go on Y gallons ?
A.: Divide 1 into X and multiply by Y.

This Subject cannot be accused of not understanding the

problem if to understand is to recognize the type by some means, find the method and apply it. It is evident however that there is not complete understanding of the problem or it would be solved directly as an example of multiplication not as one of proportion.

Similarly, to the problem " If you know how far a car goes on 1 gallon of gasolene, how may you find how far it will go on any number of gallons" the reply of Subject 23 was, " Multiply how far it goes by the number of gallons and divide by 1".

Subject 33, Problem,"The average rate of a train is such that it travels 135 miles in 4 1/2 hours. How far does it go in 3 1/4 hours?". The solution offered was :

" Required - Number of miles it will go in 3 1/4 hours
In 4 1/2 hours it goes 135 miles
In 1 hr. it goes 4 1/4 ÷ 135 = 1/30 of a mile
In 3 1/4 hrs 1/30 x 13/4 = 13/120."

One week later this same Subject was given a similar problem, " The average rate of a train is such that it travels 280 miles in 7 hours and 30 minutes. How far will it go in 3 hours and 15 minutes?". The solution given was :

" Required - How far it will go in 3 hrs and 1/4
In 7 1/2 hrs it goes 280 miles
In 1 hr it goes 15/2 x 1/280 = 15/560 miles
In 3 1/4 hrs it will go 13/4 x 15/560
= 195/2240 = 59/448 miles."

This Subject has one set way of dealing with situations of this sort and even the wording of his solutions varies by scarcely a word from week to week. This Subject is hampered in solving problems by an over-respect for stereotyped solutions which are demanded by the teacher of his grade.

C. 20. Correct Procedure Obtained from some Cue.

Subject 8 was asked, " If you know the cost of several pounds, how do you find the cost of 1 pound? ". He replied, "Divide". This Subject could not state what the divisor was to be. From the phrase ' Find the cost of 1 pound' he concluded that it was a division of some sort. The question, " What is the cost of Y pounds of candy at Z cents a pound?" brought the reply, " Multiply Y pounds by Z pounds". The answer here betrays a lack of comprehension of the real problem. This Subject is concerned primarily with operations demanded and not with reasons for the suitability of any given operation. These two problems were in familiar word settings which the Subject recognized as associated with certain specific types of operations.

C. 21. Relationship is Understood but it Cannot be put in Logical Form.

Sometimes a pupil will identify a correct operation and see the relationships involved and yet be unable to put the thought in logical form. Subject 1, knowing the cost of 1 quart of milk to be 12 cents, said that to find the cost of 1 pint you should subtract 6 from 12. Subject 13, when trying to find the number of hats that could be bought for $24, proceeded thus :

"1 hat costs $3.00
8 hats cost 8 x $3.00 = $24.00 "

This same Subject found mentally that the answer to problem 8 was 6 hours. Later when he tried to give a written solution he could get only 85 - 25 = 60. He then said, " I don't know what kind of question it is, multiplying or ---".

The recognition of a method supplemented by a faulty generalization prompted Subject 9 to reply, " Subtract the small number from the big one" when asked to find how many pages there were to be read yet when the number in a book and the number already read were known.

When asking when to add or multiply two numbers found in a problem, it was discovered that many pupils could identify the correct operation when its need arose but they could not tell what were the relations existing among the items of the problem.

C. 22. Wrong Use of the Equation

The improper use of equations is not an error in the sense that it necessarily leads to a wrong answer. It is a weakness in the supervision of arithmetic that pupils should be allowed to use incorrect statements throughout their work. Inaccurate statements foster inaccuracy of thinking. The following examples are typical of those errors in the use of equations found in this study.

Subject	Problem	Use of Equation
A	12	The rug is 12 x 9 = 108 square feet = 108 ÷ 9 = 12 sq.yds = \$1.38 x 12 = \$16.56.
B	7	85 - 25 = 60 ÷ 6 = 10.
C	8	She has read 25 = 85 - 25 = 60 pages
D	12	Cost = 9 x 12 sq.ft. = 108 sq. ft. = 12 x \$1.38 = \$16.56.

E	12	Cost of linoleum = 9 x 12 sq, ft. $\frac{9 \times 12 \times 138}{9} = \$16.56$.
F	12	Area = $\frac{9 \times 12 \times 138}{9} = \$16.56$

Four of these six pupils were from the one class of grade Vll pupils. It may be that someone has failed in their supervision to note and correct this particular type of error. From observation we know however that such errors ~~not~~ are not considered to be very serious by many teachers of the junior grades.

C. 23 Formulae or Necessary Facts not Known

All problems involving the denominate numbers and those concerned with mensuration are likely to call for the application of certain memorized facts or formulae.

In this investigation a quart was thought to be at times the equivalent of 4 pints, the area of a square was found by squaring the perimeter, the length of one side of a square was found by taking 1/2 of the perimeter, and in problem 13 different Subjects tried to find the volume of a sphere by the use of the formulae below given.

$V = r^3 = 12 \times 12 \times 12 = 1728$
$V = 22/7\ r^3$
$V = 22/7\ r$
$V = 3 \times 2 \times 22/7 \times r^2$
$V = 4 \times 22/7 \times r^3$

There is generally little hope of deducing the correct formula if it has been forgotten. There are cues that should assist one at times. For example, Subject 6 (Grade lV)

was given the problem of finding the volume of a sphere with known radius. He could not begin a written solution although in the questioning he said, " If I am to find the volume it must be the cube of something.

No argument is needed to convince the reader that to solve problems efficiently one must have many memorized facts at his command.

C. 24 Errors Connected with Denominate Numbers

The errors that group themselves under this heading may be classified into seven groups, (1) Those where the wrong facts from the denominate tables were available, (2) Those where the wrong labels were attached to the numbers but where the reason for the mistake could not be ascertained, (3) Those where the individual facts were known but where the relation of unit to unit was not known, (4) Those where the fact was given correctly but where it could not be applied correctly, (5) Those where the wrong concept was held for a given measure, (6) Those where the concept was merely incomplete, and (7) Those where certain facts were memorized but where the actual meaning and practical value of the same were quite faulty. Examples of each of these types of error follow:

Type	Subject	Problem	Error
1	M	12	5 1/2 sq.ft. =1 sq. yd.
	11	6	1 quart = 4 pints If 1 quart costs 14 cents, 1 pint will cost 1/4 of 14 cents.
2	16	12	108 sq. ft. ÷ 9 = 12 yards.

3	11	6	1 quart = 1/2 pint
	11	6	14 cents for 1 quart = 56 cents for 1 pint
4	12	11	There are 60 pounds in 1 bushel of wheat. 2520 pounds are equal to 2520 x 60 = 15120 bushels
5	23	10	The perimeter of the field is 60 ft. The area is 60 ÷ 4 = 15 sq. ft.
	18	10	The perimeter is 60 sq, ft.
	10	12	The perimeter is 42 feet= 42÷3 = 14 sq. yds.
	28	10	Estimated the size of 1 sq. ft. to be 22 inches, a length.
6	15	10	One side is 15 feet Area is 15 x 15 = 225 feet
7	7	2	In indicating a town or city that was 20 miles distant he chose one that was 215 miles away.
	17	10	Estimated that a room 18 feet long was more than 60 feet in length.
	16	11	Estimated that a bushel measure was over four times its actual size.
	23	5	Estimate of a box that would hold 7 pounds of candy was large enough to hold at least 20 pounds.
	14	12	A diagram of 1 sq. yd. as drawn by the Subject was 1/36 of its true size.

The following is an example of work unaccompanied by meaning of the denominate units involved. Subject 10 in problem 12 offered the following:

$$\begin{array}{r} 12 \\ \underline{\text{x}\ \ 2} \\ 24 \\ \underline{18} \\ 42 \end{array} \qquad \begin{array}{r} 9 \\ \underline{\text{x}\ 2} \\ 18 \end{array} \qquad \begin{array}{rl} 3\overline{)\ 42} & \\ 14 & \text{sq.yd.} \\ \underline{\text{x}\ \ 4} & \\ 56 & \text{sq. yd.} \end{array} \qquad \begin{array}{r} \$1.38 \\ \underline{56} \\ 828 \\ \underline{690\ \ } \\ \$77.28 \end{array}$$

The pupil found the perimeter in feet and attempted to change this measure to square yards by dividing by 3. When asked why 14 was multiplied by 4 he replied, " There are 14 square yards and each square yard has four sides". Twice

in this work the pupil used the idea that a square yard is part of the total perimeter. Some light was thrown upon the cause of the error in thinking when after being asked what a square yard was, he replied, " It is the same distance all around". The habit of thinking of 'square yard' in incorrect terms was so fixed that when, ten minutes later, a similar question was given, where the sides were 70 feet and 68 feet, he said, " To find square feet add them and multiply by two". This error was made in spite of the fact that in the original problem the Subject had been required to divide the rectangle into its 108 constituent squares.

No attempt has been made in the above classification to give the relative frequency of the errors, as the frequency is merely dependent upon the particular questions that chanced to be asked. There was no attempt to control the frequency of any question and therefore a question bears in its frequency no relation to its actual importance in the thinking of the Subjects.

C. 25 Confused by the Details of Writing

Subject 1 in problem 4 read the problem aloud and then siad, " That's some question. That's a hard one. There are 80 and 94 seats took up. Do you want me to find how many seats are left? There were 220 seats and you took 80 and 94 away." At this point the Subject wrote down 90 as a beginning for the solution. He stopped there and wrote no more.

Q.: Why have you stopped?
A.: I can't do it.

The complete diagnosis showed that the Subject was unable to give the written solution although he was able to give, (a) a complete outline of what was required to be done, and, (b) correct answers for both bits of mechanical work involved in the problem. When given the two examples

$$\begin{array}{r} 80 \\ \underline{+94} \end{array} \quad \text{and} \quad \begin{array}{r} 220 \\ \underline{-174} \end{array} \qquad \text{he wrote} \quad \begin{array}{r} 80 \\ \underline{+94} \\ 174 \end{array} \qquad \begin{array}{r} 220 \\ \underline{-\ 174} \\ 46 \end{array} .$$

C. 26 Memory Contributes Wrong Items

The errors of this type were of four classes, (1) The wording of the problem had been forgotten with the result that the knowns and unknowns became alternated, (2) The main features of the problem were correctly remembered but the quantitative terms became attached to the wrong units, (3) In the course of the solution, after the problem had been clearly in mind, the Subject forgot the course that had been begun and attached the wrong labels to certain of the quantities, and (4) Wrong items which had nothing to do with the problem were contributed. Single examples of these errors are given below.

Type	Subject	Problem	Error
1	16	6	Subject said to divide by 2 to find how many pints there are in a given number of quarts. The pupil knew the correct method but he forgot that he was changing quarts to pints and not pints to quarts.
2	17	4	Subtracted Y from Z instead of Z from Y. The problem read, " If there were Y seats on the train and Z of them were taken, how many would be left?"

3	14	7	After working the problem correctly and arriving at the quantitative term 8, the Subject called it 8 dollars instead of 8 hats.
4	26	11	The response " One ton" was given to the stimulus '2520 pounds'.

C. 27 Led Astray by False Cues

There were 14 pupils of grades 111 and 1V who tried problem 3 and of this number four added $3.28 to $4.95 to get the answer. It has been pointed out already that the wording of the problem caused errors at times. The words 'how much more' acted as a cue for addition. The example exemplifies the extent to which many pupils are dependent upon supplementary aids.

Sometimes as a pupil proceeded with his solution he allowed some part of the written content to act as a cue to a wrong process. For example, Subject 12 in working problem 8, gave the following solution:

" 85 pages in the book
25 has read
60 minutes in an hour
x 10
600 more pages to read. "

The explanatory notes accompanying the mechanical work were given orally after the solution was completed. The word 'hour' appeared in the problem and when the number 60 appeared in the written work, the Subject accepted the suggestion that this 60 meant 60 minutes in an hour.

Another Subject made this same sort of error in dealing with the speed of a train. The word 'hour' appeared in the

problem and the large number of miles given was taken as minutes that were to be divided by 60 to be changed to hours.

In a few instances the Subjects allowed the writer to influence them by the questions that were intended to test the understanding of principles. Wrong suggestions were conveyed by certain questions. Concerning problem 6, Subjects 4 and 5 were asked, " Would you add 2 to 14 because there are 2 pints in 1 quart?" and both replied, "Yes".

C. 28 Difficulties with Reversed Order of Operations

The errors under this heading show themselves in both the problem-solving and in the answers to the general questions. When in problem 7 Subject 13 wrote, " 1 hat costs $3.00, 8 hats cost 8 x $3.00 = $24.00 " , presenting an example in multiplication instead of one in division, there was shown a lack of appreciation of the order of the operation. The logic of arithmetic is here at fault. When several Subjects wrote

$$\begin{array}{r} 328 \\ -\ 495 \\ \hline 167 \end{array} \qquad \begin{array}{r} 6 \\ -10 \\ \hline 4 \end{array} \qquad \text{or} \qquad \begin{array}{r} 174 \\ 220 \\ \hline 46 \end{array}$$

it was because the relationships existing among the numbers were not understood.

Sometimes the general questions drew responses that were influenced by the difficulty here under discussion. Subject 1 said that 6 plus 10 was not equal to 10 plus 6; Subject 5 said that 10 plus 6 was not as big as 6 plus 10, his mind being occupied with the second number of each pair;

Subject 7 thought that 220 could be subtracted from 174. When asked " When do you subtract two numbers that you find in a problem?" Subject 11 replied, " To find what is left. Subtract either the small one from the big one or the big one from the small one".

After surveying these facts, one wonders whether children should so frequently be given rules of manipulation when they know so little about the thing they are doing and when they appreciate as little as they do the relation of one number to the other, either as numbers in a number system or as two items in a complex relationship set by the conditions of the problem.

C. 29 Meagre Knowledge of the Number System

In many instances pupils get wrong results in the mechanical work that could be avoided if they were somehow conscious of the magnitude of the numbers involved. Subject 21 said that if there were 220 seats available and 174 persons were seated, there would be 146 seats left empty. True this is an error in subtraction but we are interested in the thought processes that accompany the subtraction in this problem-setting. The Subject is either not imaging the problem situation or he has such an undeveloped number sense that flagrant inconsistencies remain unnoticed.

Of the same type is the error of Subject 1 in the mechanical work of problem 3. His work " $\begin{array}{r}495\\ \underline{-328}\\ 1176\end{array}$ " suggests that

he is deficient in knowledge of subtraction. Two supplementary thoughts may be added. One is that pupils make errors of this sort even when they can do the mechanical work if set down in the ordinary drill-example fashion; the other is that these same errors would not pass unnoticed if the pupil's thinking concerning the number system were mature. That 300 subtracted from less than 500 should be thought to give a remainder of over 1100 is surprising.

The deficiency under discussion revealed itself in another manner in the work of Subject 10, problem 11. When wishing to obtain the number of 60's in 2520 the Subject proceeded as follows:

$$\text{"}\begin{array}{r}60\\ \underline{\times\ 2}\\ 120\end{array}\qquad\begin{array}{r}60\\ \underline{\times\ 4}\\ 240\end{array}\qquad\begin{array}{r}60\\ \underline{\times\ 6}\\ 360\end{array}\qquad\qquad\begin{array}{r}60\underline{)\ 2520\ }\\ 42\end{array}\quad\text{"}$$

indicating that he carried through three multiplication steps before realizing that 2520 was a number so large that this process of trial multiplication would be wasteful of time.

Subject 3 in problem 5 estimated that the answer should be about 45 cents. The fact that 1 pound cost 35 cents and that but 10 cents more is included as the cost of 7 pounds, shows a lack of appreciation of the magnitude of numbers. A ten could replace several thirty-fives under no other considerations.

Subject 12 in problem 5 estimated the answer to be less than 25 cents (24 cents). Why? Probably because some wrong

mental calculation ground out the answer 24 instead of 245. This mistake is a mere accident. The more important fact is that a pupil in grade V should not feel the impossibility of the situation. The associations between 'ten', 'thirty' and 'hundred' may be very weak.

Subject 4 worked the examples 80 + 94 = 174 and 220 - 174 = 46 showing that he could do the mechanical work, and yet when these same facts were in problem setting, the 80 was added to the 94 by placing the numbers side by side as 8094. Proceeding then to the subtraction step, he gave as his work "

$$\begin{array}{r} 200 \\ -8094 \\ \hline 8106 \end{array}$$

" .

Subject 7 in problem 30 offered the original example

$$\begin{array}{r} 228 \\ -495 \\ \hline 33 \end{array}$$

but when asked to subtract 328 from 495 he put down the following:

$$\begin{array}{r} 495 \\ 328 \\ \hline 167. \end{array}$$

Such a deficiency as the above is not revealed to the teacher when the child is given sums to do or when he is engrossed in drill examples. The real deficiency is in the thinking about numbers, their magnitudes and their relations, and not in any faulty mastery of some rule of subtraction that concerns itself with borrowing from the ten's column to bring over to the unit's column.

One Subject subtracted 176 from 220 and got 46 when the work was down in drill form, but when she was given the responsibility of arranging the numbers herself, her subtract-

ion example appeared in this form:

$$\begin{array}{r} 346 \\ -19 \\ \hline 156. \end{array}$$

This type of work accompanies vague notions about the notation system. Individual errors in mechanical work are single in number and they do not recurr unless there lies behind some deficiency that constantly affects the work. There is evidence here that some pupils have such a vague idea concerning the number system that a constant factor is at work causing errors whenever they manipulate numbers.

C. 30 Items Improperly Labelled.

Sometimes pupils attach the wrong labels to the numbers which they have obtained in their solutions. Subject 36 performed the following multiplication making the error mentioned :

$$\begin{array}{rl} 112320 & \text{lbs} \\ 6 & \text{cents a lb} \\ \hline 673920 & \text{pounds} \quad . \end{array}$$

Subject 31 in working a problem about the average rate ot trains treated 356 2/5 miles as follows:

356 2/5 miles distant in exactly 9 hrs

$$\frac{1782 \times 9}{5} = 16038 \text{ minutes.}$$

having attached the wrong label the next step was 16038 minutes = 16038 ÷ 60 = ---- hours.

C. 31 Errors in Copying Digits from Step to Step in the Problem

The simplest errors here were mis-readings and wrong transferring of single digits. Subject 13 wrote :

" 1 bushel costs \$1.40, 42 cost \$1.40 x 40 = \$56.00".

Subject 18 found the correct number of bushels in problem 11 and then made an error in the price per bushel :

$$\begin{array}{r}\$1.46\\ \underline{\times\ \ 42}\end{array}\quad \text{------.}$$

An error in copying due to poor arrangement of the written work was found in the solution given bu Subject 26:

" Area = 18 x 14 = 9) 252 ft ÷ 9 = ( ) sq. yds.
1 yard costs $1.36
9 yards cost 9 x $1.36 = $12.24 ".

If this solution had been written in straight-forward fashion no such error would have entered.

C. 32 Evaluation of Results

In this investigation there was but one instance when a Subject upon completing a bit of work, turned voluntarily to check the accuracy of the results. It may be that at times some mental checking took place but this went unrecorded. In one instance a Subject subtracted 174 from 220 and then reversed the process to test the accuracy of the work.

After the first written solution was completed, each Subject was asked to evaluate the result he had obtained. The responses given and the attempted evaluations reflect the ability and the inclinations of the pupils to evaluate their answers when the question of accuracy is raised by another individual. It was evident that these Subjects were not trained in evaluating arithmetical work. Their attempts to make corrections were clumsy. In many instances

the Subjects were quite unprepared to do anything. Pupil 4 said the result was quite correct although in problem 3 he had added the $3.28 to the $4.95. That he was unprepared for evaluating the result was shown by the fact that the solution had not been preceded by an analysis of the problem. He said that 'how much more ---' meant how much more than $4.95.

Some pupils when challenged give a hurried glance over the mechanical work which they recorded on the paper and then state that the results are quite correct. This was what Subject 13 did for problem 8 although he had an answer of 40 pages when the problem asked him to find how many hours. Some Subjects when questioned thought at once that their results were wrong and hastened to make some change which did not prove to be a correction. Subject 14 got $2.88 as the result for one problem. When she saw that her work was even questioned, she said, " Oh, No, $288.00". The numbers in the problem were of the order of the twenties and the thirties and there was nothing in the setting to suggest any such number as 288. One Subject did problem 6 correctly. When asked to evaluate the result he said, " No, I found a mistake. You should multiply by 2 instead of divide by 2". He accordingly changed his work to the new and incorrect form.

In a few instances the Subjects when challenged sensed the practical situation, which up to that time had not been

thought of. One Subject did this when his answer of $4.90 for linoleum for a floor was questioned. He said that he did not think that $5.00 would buy enough linoleum for the floor.

In all the replies only one example of real checking of processes was apparent. One adult Subject worked problem 13 about the volume of a sphere. When asked to evaluate his result he tried to compare the valome of a sphere with the volume of a cube that might be cut from the sphere.

In order to summarize the general responses to the demand for evaluation, 114 consecutive replies were tabulated. Of these 100 merely expressed an opinion with respect to the work that had been done, while 14 resulted in some activity toward the problem. Of the 100 rather worthless evaluations 84 were "Yes", 2 were "No", 5 were " I think it's wrong", 2 were "I don't know", 1 was " I don't think so" and 1 was " It might be wrong". Five Subjects threw the onus of proof upon the investigator. Of the 14 Subjects who turned to the problem when challenged, 12 corrected errors in the mechanical work, and only two of them attempted to verify or test the thought processes by which the results had been obtained.

These Subjects were not trained in evaluating results. Less than 2% of them went back to verify assumptions. Is this because there is a blind acceptance of any method that happens to come to mind in association with the problem

Seventy-four percent of the Subjects, many of whom had wrong answers, replied when challenged, " Yes, This is correct" or they gave answers of like meaning. Evidently many pupils do not search for solutions; they do not 'solve' problems; they have fixed modes of behaving towards certain problem situations.

ERRORS THAT DEAL WITH GENERALIZATIONS

D. 33 Specific Examples are Solved but Generalizations are not Grasped.

Subjects 1, 2, 4, 5, 24 and 3 all gave a correct solution for problem 1 on the first trial. Their answers to the question, " If you knew the ages of two boys, how could you find how much the one was older than the other?" were as follows:

Subject	Reply
1	The big one would likely be the oldest.
2	By putting them down and adding them.
4	Take the most away from the other one.
5	Subtract the lower one from the higher one.
24	One looks older than the other one
3	Add them.

It is apparent that four of these Subjects can deal with only the simple concrete cases. Subject 5 has abstracted a common element from situations of this kind while Subject 4 notes the fact that the operation of subtraction fits them all, but he has not perfected his generalization.

Inability to note similar elements is seen in the work of Subject 10. In problem 12 he found the cost of the lin-

oleum correctly, but he did not succeed in finding the cost of the tile required for a floor 68 feet by 70 feet, if the tile cost $5.00 per square yard. The same deficiency was shown by Subject 2 in problem 3. She failed to subtract $3.28 from $4.95 wishing to add enough to the $3.28 to make the $4.95. When the amounts involved were changed to 10 cents and $1.00, the Subject subtracted, but upon returning to the original problem once more proceeded with her trials at addition.

Subjects showed by their answers to the oral questions different stages of progress toward generalizing ability. The Subjects were asked " How do you find the cost of any number of pounds when you know the cost of 1 pound? ". Three of the replies were:

Reply	Comment
1. By adding up the pounds	The Subject meant that the cost of the several pounds should be added.
2. Put them down and add them up	
3. Multiply the number of pounds by ---- (incomplete).	The correct process is identified and can be used but it is not clearly enought thought out to permit of definite formulation.

Progress from the slow method of equal subtractions to the idea of division is seen in the following answers to the question, " If you know the cost of several pounds, how do you find the cost of 1 pound? ": (1) Subtract it, (2) Add down that much until you get to one pound, (3) Divide the several pounds by something, (4) Divide the number of--

pounds by the cost, (5) Divide the cost by ---(incomplete), (6) Divide the total cost by the number of pounds. In (1) and (2) the idea of subtraction is uppermost, in (3) division is vaguely introduced, in (4) division, total cost and number of pounds all appear, but the latter two are not properly related, but, in (6) the generalized procedure is clearly expressed.

D. 34 A Memorized Generalization Cannot be Applied

This difficulty occurred where there were formulae which gave a very condensed form for a very great amount of information.

A simple example of this deficiency occurred frequently in problem 7. Nearly all the Subjects said, " The perimeter is the distance around the square". This is the linguistic form of the equation $P=4L$, where L is the length of one side. That often the statement was not understood is clear from the fact that the Subjects proceeded to take 1/2 of 60, or even 60 itself, as the length of one side.

Problem 13 gave an example of a very complex generalization. The formula, sometimes known but more often discovered while doing one of the special selection of methods tests, was $V = 4/3 \times 22/7 \times r^3$, where V is the volume of the sphere and r is the radius. Subjects could apply this formula to the given problem where the radius was 12 inches. They were asked to continue and to find in succession the

volumes of spheres having radii of 24, 36 and 6 inches respectively. If the generalization were not used the Subject would apply the formula to each example in turn and work out the equations $V_1 = 4/3 \times 22/7 \times 12 \times 12 \times 12$, $V_2 = 4/3 \times 22/7 \times 24 \times 24 \times 24$, $V_3 = 4/3 \times 22/7 \times 36 \times 36 \times 36$, and $V_4 = 4/3 \times 22/7 \times 6 \times 6 \times 6$. If on the other hand the Subject saw the generalization as applied to volumes V_2, V_3 and V_4, he noted that the volumes desired could be derived from V_1 by simple multiplication by 8, 27 and 1/8 respectively.

Of the 14 Subjects, whose number included high school graduates, teachers and senior year university students, only two saw and applied the generalization. One of these Subjects was an instructor in mathematics and the other was a student in honor mathematics. Subject 45 worked each example independently but said while working the third example, " There is a short way, I know, but I don't see it. There is some relationship I know".

The above problem is a difficult one, but it illustrates again that there are individual differences in the degrees to which a generalization is understood by different individuals who have memorized the words expressing the generalization. There is evidence that generalizations are of great range in the matter of difficulty. The simplest mathematical generalizations may be understood by the pupils in grade 111 (Subjects 2, 3, 4 and 5 understood some of

them), some are understood by the pupils in grade Vl, but there are others that may be understood only after the experience in mathematics that comes with years of high school and college training.

D. 35 Faulty Generalizations

To deal with recurring arithmetical situations, pupils develop or are given generalizations, which are faulty or incomplete at times and must therefore be corrected later.

Subject 15 tried to carry over to the minutes-hour situation a generalization that had served him effectively in the cents-dollar situation. When he arrived at the result '876 minutes' he changed it to 8 hours and 76 minutes, using the same method of reduction that had served him in the other situation. There are certain similiarities between the two situations but this Subject included an extra and wrong one.

Another type of error in abstraction called forth the following reasons for subtracting any two numbers that were found in a problem:

1. When one is higher than the other.
2. When you want to find something that is less.
3. To make it less.

In the pupils's experience to date, probably subtraction has meant 'to make less', but the time is coming when this generalization must be modified. In this type of error

we have an extension of a generalization, an incomplete form, where all the essentials are not grasped, but where supplementary ideas are given first place.

CHAPTER 5

CONCLUSIONS

CHAPTER 5

The preceding chapters present evidence from which the following conclusions are drawn.

1. The ability of pupils to find correct answers to problems in arithmetic is a poor measure of their actual problem-solving ability.
2. Correct answers are obtained for problems when the anapysis that should precede any manipulation of the numbers involved has not been made.
3. Many Subjects do not find correct answers to problems have just as complete an understanding of the processes involved as have those who succeed in getting the correct answers. It is true also that at times pupils obtain the correct answer and yet have little if any understanding of the methods involved.
4. Many pupils learn correct methods of manipulation without being able to explain why certain processes are carried out, and without seeing the relationships that exist among the elements of the problem.
5. Much problem-solving is but a slavish adherence to type methods of procedure which become associated with certain language and number situations that exist in the presented problem.
6. Operations with numbers generally involve some manipulation or use of the number system. There are many pupils who become skilful in manipulation without know-

ing the principles involved in the use of the system. There are other children who cannot manipulate numbers.

7. Pupils may become expert in solving individual problems without appreciating the generalization that fits the particular case.
8. Sometimes pupils have memorized generalizations that they are not able to apply. They cannot recognize the proper application when it arises.
9. The evidence is against the view that children are being trained in independence in problem-solving. There is little weighing of evidence and in few instances do pupils evaluate their results.
10. Pupils are sometimes given problems to solve when they lack the experience that leads to an understanding of the concrete setting of the problem.
11. In many instances, training in grammatical analysis of the printed problem would result in facility in finding the real problem that the words are meant to convey.

The data herein presented were gathered from a limited number of pupils. The universality of the findings in other school systems must be determined by an extension of the investigation.

It is to the analysis of the thought processes that was made throughout this study that the writer attaches the maximum importance.

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